Chalmers/Gothenburg University
Mathematical Sciences

EXAM

TMA947/MMG621
OPTIMIZATION, BASIC COURSE

Date:	16–04–05
Time:	House V, morning, 8^{30}–13^{30}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Johannes Borgqvist (ankn. 5325)
Result announced:	16–04–15 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$\begin{array}{llllllll} \text{minimize} & z = 8x_1 & +3x_2 & +4x_3 & +x_4, & & \\ \text{subject to} & 2x_1 + & x_2 & +3x_3 & - x_4 & = & 5, \\ & x_1 + & x_2 & +2x_3 & - x_4 & = & 3, \\ & x_1, & x_2, & x_3, & x_4 & \geq & 0. \end{array}$$

Instead of trying to solve the problem using phase I and phase II simplex method separately, we could solve it in "one-shot". We consider the modified problem:

$$\begin{array}{lllllllll} \text{minimize} & z = 8x_1 & +3x_2 & +4x_3 & +x_4 & +My_1 & +My_2, & & \\ \text{subject to} & 2x_1 + & x_2 & +3x_3 & -x_4 + & y_1 & & = & 5, \\ & x_1 + & x_2 & +2x_3 & -x_4 & & + \quad y_2 & = & 3, \\ & x_1, & x_2, & x_3, & x_4, & y_1, & y_2 & \geq & 0, \end{array}$$

where M is a very large but unspecified number such that $a + M > 0$ and $a - M < 0$ for all real number a.

(1p) a) Is the modified problem with M always feasible? Assume that the optimal objective value of the modified problem is bounded from below. If we solve the modified problem, what can we say about the feasibility and optimal objective value of the original problem, depending on the optimal values of y_1 and y_2 in the modified problem? Explain your answers.

(2p) b) Solve the modified problem with M using the simplex method, keeping M as a unspecified large number. If the problem can be solved to optimality, write down an optimal solution and objective value of the original problem.

Aid: Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Question 2

(true or false)

The below three claims should be assessed. Are they true or false, or is it impossible to say? Provide an answer, together with a short motivation.

(1p) a) Consider a standard LP problem, for which you apply the Simplex method. Suppose also that you have used Phase I of the simplex method and identified a basic feasible solution.

Claim: Then in Phase II you will be able to identify an optimal solution to the given problem.

(1p) b) Suppose that you are solving an unconstrained optimization problem in which you minimize a differentiable function f. Suppose further that at a given vector $\boldsymbol{x}$ you have generated a descent direction $\boldsymbol{p}$.

Claim: Then the Armijo rule will provide a positive, finite step length in which the objective function has a lower value of $f : \mathbb{R}^n \to \mathbb{R}$ than at $\boldsymbol{x}$.

(1p) c) Consider the problem of minimizing a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ over a bounded polyhedral set. Suppose further that we attack this problem by utilizing the Frank–Wolfe method. Suppose then that we have solved the linear subproblem of the algorithm.

Claim: Then the linearized objective function has an optimal value in the linear subproblem that is lower than or equal to the objective value at the current iteration.

(3p) Question 3

(optimality conditions)

Farkas' Lemma can be stated as follows:

Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{b}$ an $m \times 1$ vector. Then exactly one of the systems

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b}, \\ \boldsymbol{x} &\geq \boldsymbol{0}^n, \end{aligned} \tag{I}$$

and

$$\begin{aligned} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} &\le \boldsymbol{0}^n, \\ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} &> 0, \end{aligned} \tag{II}$$

has a feasible solution, and the other system is inconsistent.

Establish Farkas' Lemma.

(3p) Question 4

(Frank–Wolfe)

Consider the problem to

$$\underset{x \in X}{\text{maximize}} \ f(\boldsymbol{x}). \tag{2}$$

Assume that X is a polyhedron and $f \in C^1$. Let $\bar{\boldsymbol{x}} \in X$ be a point to which the Frank–Wolfe algorithm converges within a finite number of iterations on the problem (2). Can we guarantee that $\bar{\boldsymbol{x}}$ is optimal? If not, which properties can we guarantee that $\bar{\boldsymbol{x}}$ has, and which additional requirements are necessary to guarantee that $\bar{\boldsymbol{x}}$ is an optimal solution to the problem (2)?

(3p) Question 5

(Lagrangian duality)

Consider the optimization problem

$$\begin{aligned} f^* := \underset{\boldsymbol{x}}{\text{infinum}} \quad & f(\boldsymbol{x}), \\ \text{subject to} \quad & \boldsymbol{g}(\boldsymbol{x}) \le \boldsymbol{0}^m, \\ & \boldsymbol{x} \in X. \end{aligned}$$

Let the Lagrange function be defined as $L(\boldsymbol{x}, \boldsymbol{\mu}) := f(\boldsymbol{x}) + \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{g}(\boldsymbol{x})$. Assume that $\boldsymbol{\mu}^*$ is a Lagrange multiplier. That is, $\boldsymbol{\mu}^* \ge \boldsymbol{0}^m$ and $\inf_{\boldsymbol{x} \in X} L(\boldsymbol{x}, \boldsymbol{\mu}^*) = f^*$. Show that $\boldsymbol{x}^*$ is optimal if and only if

$$\begin{aligned} \boldsymbol{x}^* \in X, \quad \boldsymbol{g}(\boldsymbol{x}^*) &\le \boldsymbol{0}^m, \\ \boldsymbol{x}^* &\in \underset{\boldsymbol{x} \in X}{\operatorname{argmin}} \ L(\boldsymbol{x}, \boldsymbol{\mu}^*), \\ \boldsymbol{\mu}_i^* g_i(\boldsymbol{x}^*) = 0, \quad i &= 1, \ldots, m. \end{aligned}$$

(3p) Question 6

(integer programming modeling) Let a chessboard be a $n \times n$ grid with n being some integer. A queen can move any number of squares horizontally, vertically or diagonally. See Figure. 1 for an illustration of the possible moves of a queen.

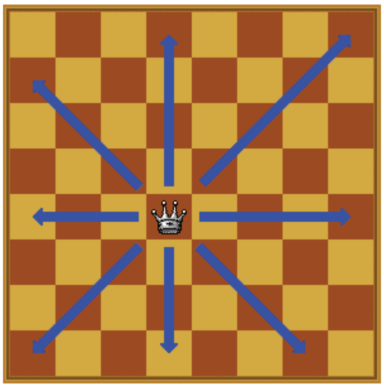

Figure 1: Possible moves of a queen. Source: `http://www.chess-poster.com`

For this problem, we can place an arbitrary number of queens on the chessboard. We are asked to find a configuration with the minimum number of queens so that

- each square either is occupied by a queen or can be attacked by a queen,
- no two queens can attack each other.

Formulate the problem to find the desirable configuration as an integer program.

(3p) Question 7

(gradient projection algorithm)

Consider the optimization problem to

$$\begin{aligned} \underset{x_1,x_2}{\text{minimize}} \quad & f(\boldsymbol{x}) := \tfrac{1}{2}(x_1 - 2)^2 + \tfrac{1}{2}(x_2 - \tfrac{3}{2})^2, \\ \text{subject to} \quad & \boldsymbol{x} \in X = \{(x_1, x_2)^{\mathrm{T}} \mid -1 \le x_i \le 1, \;\; i = 1, 2\}. \end{aligned}$$

We consider solving the problem using the gradient projection algorithm. Start with the initial point $\boldsymbol{x}^0 = (0,0)^{\mathrm{T}}$. Perform one step of the gradient projection algorithm (so that you obtain the next iterate $\boldsymbol{x}^1$). Use the projection arc and perform exact minimization line search. That is, $\boldsymbol{x}^{k+1} = \mathrm{Proj}_X[\boldsymbol{x}^k + \alpha^k \boldsymbol{p}^k]$ for the appropriate search direction $\boldsymbol{p}^k$ and step size α^k for each iteration k. Is $\boldsymbol{x}^1$ optimal or not? Explain your answer.