Chalmers/Gothenburg University
Mathematical Sciences

EXAM SOLUTION

TMA947/MAN280
OPTIMIZATION, BASIC COURSE

Date: 10–04–06
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Question 1

(the simplex method)

(2p) a) We first rewrite the problem on standard form. In the first constraint we change sign and subtract a non-negative slack (surplus) variable. The upper bound on x_2 is considered as a linear constraint, and in this constraint a second slack variable is added. We get

$$\begin{array}{llrrrrrl} \text{minimize} & z = & -2x_1 & + \; x_2, & & & & \\ \text{subject to} & & -x_1 & +3x_2 & -s_1 & & = & 3, \\ & & & x_2 & & +s_2 & = & 2, \\ & & x_1, & x_2, & s_1, & s_2 & \geq & 0. \end{array}$$

In phase I, an artifical variable, $a \geq 0$ is added in the first constraint. s_2 is used as the second basic variable. The phase I problem is

$$\begin{array}{llrrrrrrl} \text{minimize} & w = & & & & & a, & & \\ \text{subject to} & & -x_1 & +3x_2 & -s_1 & & +a & = & 3, \\ & & & x_2 & & +s_2 & & = & 2, \\ & & x_1, & x_2, & s_1, & s_2, & a & \geq & 0, \end{array}$$

and our starting BFS is $(a, s_2)^{\mathrm{T}}$. A calculation of the vector of reduced costs for the non-basic variables x_1, x_2 and s_1 gives $(1, -3, 1)^{\mathrm{T}}$, and hence, x_2 is chosen as the incoming variable. The minimum ratio test shows that a should leave the basis. Since there are no artifical variables left in the basis we have $w = 0$ which is optimal in the phase I problem and which corresponds to a BFS to the original problem. We return to the original problem using $B = (x_2, s_2)^{\mathrm{T}}$, $N = (x_1, s_1)$. The vector of reduced costs are calculated to be $(-5/3, 1/3)^{\mathrm{T}}$ and therefore x_1 is chosen as the incoming variable. The minimum ratio test shows that s_2 should be removed from the basis. Updating B to $(x_2, x_1)^{\mathrm{T}}$ and N to $(s_2, s_1)^{\mathrm{T}}$ and calculating the new reduced costs shows that $\tilde{\boldsymbol{c}}_N = (5, 2) > \boldsymbol{0}$ and hence the current basis is optimal. We have $B^{-1}\boldsymbol{b} = (2, 3)$, i.e., $\boldsymbol{x}^* = (x_1, x_2)^* = (3, 2)$, and $z^* = -4$.

(1p) b) From strong duality we have $z^* = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^* = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}^*$ where $\boldsymbol{y}^*$ is the vector of optimal dual variables. The optimal basis will not change for sufficiently small changes of β since $\tilde{\boldsymbol{c}}_N > \boldsymbol{0}$ implies that the optimal basis is unique at the current point. Therefore we have that $\frac{\partial z}{\partial \beta}$ equals the value of the dual variable corresponding to the first con straint. The expression for the dual variables is $\boldsymbol{y}^{\mathrm{T}} = \boldsymbol{c}_B^{\mathrm{T}} B^{-1}$ (which, if one does not remember is clear from the fact that $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^* = \boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{x}_B^* = \boldsymbol{c}_B^{\mathrm{T}} B^{-1}\boldsymbol{b} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}^*$). We have $\boldsymbol{c}_B^{\mathrm{T}} B^{-1} = (2, -5)^{\mathrm{T}}$, and therefore the marginal change of z^* is twice the change of β.

Question 2

(implications in theorems)

(1p) a) Any LP with $\boldsymbol{c} = \mathbf{0}$ would do (as long as the feasible polyhedron has a non-empty relative interior). Here all feasible points are optimal, but not all of them are extreme points. Another simple example is given by $\min x_1$ s.t. $0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 1$. Here, $(0, \frac{1}{2})^\mathrm{T}$ is an optimal solution, but it is not extreme since, e.g., $(0, \frac{1}{2})^\mathrm{T} = \frac{1}{2}(0,0)^\mathrm{T} + \frac{1}{2}(0,1)^\mathrm{T}$.

(1p) b) Let $n = 1$ and $f(x) = x^4$. Here, $x^* = 0$ is clearly a strict local minimum (and also the global minimum), however the hessian $\nabla^2 f(\boldsymbol{x}^*)$ is not positive definite but only positive semi-definite ($\nabla^2 f(\boldsymbol{x}^*) = 0$).

(1p) c) Let, e.g., $f(x) = -x^2$ and $S = [0, 1]$ for a minimization problem. Here, $x = 0$ fulfills the variational inequality ($\nabla f(\boldsymbol{x}^*)^\mathrm{T}(\boldsymbol{x} - \boldsymbol{x}^*) \geq 0,\ \forall \boldsymbol{x} \in S$), but it is not locally optimal.

Question 3

(modeling)

The model uses the following constants: k_c, k_r, k, T, T_O. The model uses the following variables: D - cylinder diameter, H - cylinder height. The surface area of the cylinder can be expressed as

$$H\pi D + 2\pi \left(\frac{D}{2}\right)^2 = \pi H D + \frac{\pi}{2} D^2.$$

The volume of the cylinder can be expressed as

$$H\pi \left(\frac{D}{2}\right)^2 = \frac{\pi}{2} H D^2.$$

To fit the cylinder inside a sphere, we should place the center of the cylinder at the center of the sphere. The point inside the cylinder most distant from the center is then $\sqrt{(H/2)^2 + (D/2)^2}$ length units away. This must thus be smaller than the radius R. The full model is then to

$$\text{minimize} \quad \left(k_c(T - T_O) + k_r(T^4 - T_O^4)\right)\left(\pi H D + \frac{\pi}{2} D^2\right)$$

$$\begin{aligned} \text{subject to} \quad k(T - T_O)\frac{\pi}{2}HD^2 &\geq Q', \\ \sqrt{(H/2)^2 + (D/2)^2} &\leq R, \\ H, D &\geq 0. \end{aligned}$$

The model is a nonlinear programming model.

Question 4

(linear programming duality)

(1p) a) The linear programming dual problem is to

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{\alpha} + \boldsymbol{l}^{\mathrm{T}}\boldsymbol{\beta} - \boldsymbol{u}^{\mathrm{T}}\boldsymbol{\gamma}, \\ \text{subject to} \quad & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\alpha} + \boldsymbol{\beta} - \boldsymbol{\gamma} = \boldsymbol{c}, \\ & \boldsymbol{\beta}, \boldsymbol{\gamma} \geq \boldsymbol{0}^n, \end{aligned}$$

where $\boldsymbol{\alpha} \in \mathbb{R}^m$ is the vector of dual variables for the linear constrants, and $\boldsymbol{\beta} \in \mathbb{R}^n$ and $\boldsymbol{\gamma} \in \mathbb{R}^n$ respectively are the vector of dual variables for the lower and upper bounds on $\boldsymbol{x}$.

(1p) b) Set, for example, $\boldsymbol{\alpha} \in \boldsymbol{0}^m$. Then, study the sign of each element of the vector $\boldsymbol{c}$: if $c_j = 0$, set $\beta_j = \gamma_j = 0$; if $c_j > 0$, set $\beta_j = c_j$ and $\gamma_j = 0$; finally, if if $c_j < 0$, set $\beta_j = 0$ and $\gamma_j = -c_j$. This then constitutes a feasible solution to the linear programming dual problem.

(1p) c) The conclusion is that the primal problem has a finite optimal solution; see Theorem 10.6, for example.

Question 5

(Newton's method)

(1p) a) Newton's equation:

$$x_{k+1} = x_k - \frac{\alpha x^{\alpha-1} - \exp(x)}{\alpha(\alpha-1)x^{\alpha-2} - \exp(x)}.$$

(1p) b) Probably the simplest counter-example is obtained by taking $x_0 = 1, \alpha = 2$. These initial values cause the Newton's method to generate an oscillating sequence $x_{2k-1} = 0, x_{2k} = 1, k = 1, 2, \ldots$.

(1p) c) The objective function of the problem is not convex in general [may be verified by analyzing the sign of the Hessian $\alpha(\alpha - 1)x^{\alpha-2} - \exp(x)$]. Since the convergence of the Newton method is local in nature, the method is most likely to converge to the nearest local minimum (or maximum if the hessian is negative definite). The engineer thus wrongly assumes the global convergence of the Newton method on non convex functions.

(3p) Question 6

(fundamental theorem of global optimality)

See Theorem 4.3 in The Book.

Question 7

(optimality conditions)

(1p) a) With $\boldsymbol{z} = (2, 3/2)^{\mathrm{T}}$, the appropriate problem to solve is that to

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{z}\|^2, \\ \text{subject to} \quad & x_1 + x_2 \leq 3/2, \\ & x_j \geq 0, \qquad j = 1, 2. \end{aligned}$$

The objective function is strictly convex: $\nabla f(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{z}$ and $\nabla^2 f(\boldsymbol{x}) = \boldsymbol{I}^n$, where $\boldsymbol{I}^n$ is the identity matrix, so the Hessian $\nabla^2 f(\boldsymbol{x})$ matrix is positive definite everywhere. The problem is a convex one, since also the feasible set is convex—it is indeed a polyhedron.

(1p) b) Changing sign of the second group of constraints, and introducing the Lagrange multiplier vector $\boldsymbol{\mu} \in \mathbb{R}^3$, we obtain the KKT conditions for a feasible vector $\boldsymbol{x}^*$ as follows:

$$\begin{aligned}\boldsymbol{x}^* - \boldsymbol{z} + \mu_1 \begin{pmatrix}1\\1\end{pmatrix} + \mu_2 \begin{pmatrix}-1\\0\end{pmatrix} + \mu_3 \begin{pmatrix}0\\-1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix},\\ \mu_1(x_1^* + x_2^* - 3/2) = 0,\\ \mu_2 x_1^* = 0,\\ \mu_3 x_2^* = 0.\end{aligned}$$

As the constraints of the problem in a) are affine, the Abadie constraint qualification (CQ) is satisfied; therefore, the KKT conditions are necessary for a local minimum at $\boldsymbol{x}^*$.

As was established in a) above, the optimization problem is a convex one. The above KKT conditions then are sufficient for a feasible vector $\boldsymbol{x}^*$ to be a global minimum of the above problem.

(1p) c) At the given vector $\boldsymbol{x} = (1, 1/2)^{\mathrm{T}}$, it is clear that at any KKT point, $\mu_2 = \mu_3 = 0$ must hold, while complementarity leaves μ_1 free. The remaining linear equation becomes:

$$\mu_1 \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix};$$

hence, $\mu_1 = 1$. The KKT conditions are satisfied; the vector $\mu^* = (1, 0, 0)^{\mathrm{T}}$ is a vector of Lagrange multipliers, corresponding to the optimal solution $\boldsymbol{x}^* = (1, 1/2)^{\mathrm{T}}$.