Chalmers/Gothenburg University
Mathematical Sciences

EXAM SOLUTION

TMA947/MAN280
OPTIMIZATION, BASIC COURSE

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Examiner: Michael Patriksson

Question 1

(the simplex method)

(2p) a) We first rewrite the problem on standard form. We multiply the objective by -1 to obtain a minimization problem and introduce the variables x_2^+ and x_2^- such that $x_2 = x_2^+ - x_2^-$, and slack variables s_1 and s_2.

$$\begin{array}{llrrrrrl} \text{minimize} & z = & -3x_1 & -x_2^+ & +x_2^- & & & \\ \text{subject to} & & 3x_1 & +2x_2^+ & -2x_2^- & -s_1 & & = 1 \\ & & 2x_1 & +x_2^+ & -x_2^- & & +s_2 & = 2 \\ & & x_1, & x_2^+, & x_2^-, & s_1, & s_2 & \geq 0. \end{array}$$

In phase I the artificial variable a is added in the first constraint, s_2 is used as the second basic variable. We obtain the problem

$$\begin{array}{llrrrrrrl} \text{minimize} & w = & & & & & & a & \\ \text{subject to} & & 3x_1 & +2x_2^+ & -2x_2^- & -s_1 & & +a & = 1 \\ & & 2x_1 & +x_2^+ & -x_2^- & & +s_2 & & = 2 \\ & & x_1, & x_2^+, & x_2^-, & s_1, & s_2 & a & \geq 0. \end{array}$$

The starting BFS is thus $(a, s_2)^{\mathrm{T}}$. Calculating the vector of reduced costs for the non-basic variables x_1, x_2^+, x_2^- and s_1 yields $(-3, -2, 2, 1)^{\mathrm{T}}$. Thus x_1 enters the basis. The minimum ratio test shows that a should leave the basis. We thus have a BFS without artificial variables, and may proceed with pha se II.

We have the basic variables (x_1, s_2). The vector of reduced costs for the non-basic variables x_2^+, x_2^- and s_1 is $(1, -1, -1)$. We may choose either x_2^- or s_1 to enter the basis. We take x_2^-. The minimum ratio test implies that s_2 must leave the basis. We now have x_1, x_2^- as basic variables. The vector of reduced costs for the non-basic variables x_2^+, s_1, s_2 is $(0, 1, 3)^{\mathrm{T}}$. The current point is optimal. We thus have $(x_1, x_2^-, x_2^+, s_1, s_2) = (3, 4, 0, 0, 0)$, or in the original variables, $(x_1, x_2) = (3, -4)$.

(1p) b) The reduced costs are not strictly positive; we can thus not conclude that there is a unique optimal solution. We may introduce x_2^+ into the basis; the minimum ratio test can however not provide a variable that leaves the basis (all entries are negative in $B^{-1}N_j$). This is because we may let $x_2^+ = \alpha$, $x_2^- = 4 + \alpha$ for all $\alpha \geq 0$ and obtain an optimal solution in the problem written on standard form. All these solutions however correspond to the same solution $(x_1, x_2) = (3, -4)$ in the original problem. The solution in the original problem is unique (which can also be realized by checking that it is the only KKT point).

Question 2

(optimality conditions)

(2p) a) Thanks to the linearity of the constraints, the problem satisfies the Abadie constraint qualification and the Karush–Kuhn–Tucker conditions are necessary for the local optimality of $\boldsymbol{x}^*$. As the problem is convex the KKT conditions are also sufficient for $\boldsymbol{x}^*$ to be a global optimum.

Introducing the multiplier λ for the equality constraint and $\mu_j \geq 0$ for the sign condition on x_j, we obtain the Lagrange function $L(\boldsymbol{x}, \mu, \boldsymbol{\lambda}) := -b\lambda + \sum_{j=1}^n (f_j(x_j) - [\lambda + \mu_j]x_j)$. Setting the partial derivatives of L with respect to each x_j to zero yields

$$f'(x_j^*) = \lambda^* + \mu_j^*, \qquad j = 1, \ldots, n. \tag{1}$$

Further, the complementarity conditions state that

$$\mu_j^* \cdot x_j^* = 0, \qquad j = 1, \ldots, n.$$

Together with the dual feasibility conditions that $\mu_j^* \geq 0$ for all j and that $\boldsymbol{x}^*$ fulfills the primal feasibility conditions that $\boldsymbol{x}^* \geq \boldsymbol{0}^n$ and $\sum_{j=1}^n x_j^* = b$, we have stated all the KKT conditions.

(1p) b) Suppose that the triple $(\boldsymbol{x}^*, \mu^*, \boldsymbol{\lambda}^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ is a Karush–Kuhn–Tucker point. For a j with $x_j^* > 0$ we must therefore have from (1) that $f'(x_j^*) = \lambda^*$. Suppose instead that $x_j^* = 0$. Then, since $\mu_j^* \geq 0$ must hold, we obtain from (1) that $f'(x_j^*) = \lambda^* + \mu_j^* \geq \lambda^*$, and we are done.

Question 3

(modeling)

(1p) a) Introduce the variable x_{ij} for the amount of money person i gives to person

j. The model is to

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}, & \\ \text{subject to} & \displaystyle d_i + \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = \frac{1}{n}\sum_{j=1}^{n} d_j, & i = 1,\dots,n. \\ & x_{ij} \geq 0 & i = 1,\dots,n,\ j = 1,\dots,n. \end{array}$$

(2p) b) Introduce the variables y_{ij}, where

$$y_{ij} = \begin{cases} 1 & \text{if person } i \text{ gives any money to person } j \\ 0 & \text{otherwise.} \end{cases}$$

Then the model is to

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}, & \\ \text{subject to} & \displaystyle d_i + \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = \frac{1}{n}\sum_{j=1}^{n} d_j, & i = 1,\dots,n, \\ & \displaystyle\sum_{j=1}^{n} y_{ij} = 1, & i = 1,\dots,n, \\ & x_{ij} \leq M y_{ij}, & i = 1,\dots,n,\ j = 1,\dots,n, \\ & x_{ij} \geq 0, & i = 1,\dots,n,\ j = 1,\dots,n, \\ & y_{ij} \in \{0,1\}, & i = 1,\dots,n,\ j = 1,\dots,n, \end{array}$$

where M is some large number. $M = \sum_{i=1}^{n} d_i$ is large enough.

(3p) Question 4

(the Frank-Wolfe method)

Iteration 1: $\boldsymbol{x}_0 = (0,0)^{\mathrm{T}}$ is feasible and $f(\boldsymbol{x}_0) = 0$, so we get: $[LBD, UBD] = (-\infty, 0]$. $\nabla f(\boldsymbol{x}_0) = (-3,-6)^{\mathrm{T}}$ and the solution to the LP $\min_{\boldsymbol{y}} \nabla f(\boldsymbol{x}_0)^{\mathrm{T}}\boldsymbol{y}$ is obtained at $\boldsymbol{y}_0 = (2,2)^{\mathrm{T}}$. Since f is convex, $g(\boldsymbol{y}) := f(\boldsymbol{x}_0) + \nabla f(\boldsymbol{x}_0)^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x}_0) \leq f(\boldsymbol{y})$ for all $\boldsymbol{y} \in \mathbb{R}^{\not{\vDash}}$. The LP problem is a relaxation of the original problem, hence an optimal objective value gives a lower bound. The optimal objective value of the LP is $f(\boldsymbol{x}_0) + \nabla f(\boldsymbol{x}_0)^{\mathrm{T}}(\boldsymbol{y}_0 - \boldsymbol{x}_0) = 0 + (-3,-6)^{\mathrm{T}}(2,2) = -18$. Hence, $[LBD, UBD] = [-18, 0]$. The search direction is $\boldsymbol{p}_0 = \boldsymbol{y}_0 - \boldsymbol{x}_0 = (2,2)^{\mathrm{T}}$. Line search: $\phi(\alpha) := f(\boldsymbol{x}_0 + \alpha\boldsymbol{p}_0) = f((2\alpha, 2\alpha)^{\mathrm{T}}) = 12\alpha^2 - 18\alpha$. $\phi'(\alpha) = 24\alpha - 18 = 0 \Rightarrow \alpha = 3/4 < 1$. Hence, $\boldsymbol{x}_1 = (3/2, 3/2)^{\mathrm{T}}$.

Iteration 2: $f(\boldsymbol{x}_1) = -27/4$, so $[LBD, UBD] = [-18, -27/4]$. $\nabla f(\boldsymbol{x}_1) = (3/2, -3/2)^{\mathrm{T}}$ and the solution to the LP $\min_{\boldsymbol{y}} \nabla f(\boldsymbol{x}_1)^{\mathrm{T}}\boldsymbol{y}$ is obtained at $\boldsymbol{y}_1 =$

$(1,2)^{\mathrm{T}}$. $f(\boldsymbol{x}_1) + \nabla f(\boldsymbol{x}_1)^{\mathrm{T}}(\boldsymbol{y}_1 - \boldsymbol{x}_1) = -33/4$, so $[LBD, UBD] = [-33/4, -27/4]$. The search direction is $\boldsymbol{p}_1 = \boldsymbol{y}_1 - \boldsymbol{x}_1 = (-1/2, 1/2)^{\mathrm{T}}$. Line search, $\phi(\alpha) := f(\boldsymbol{x}_1 + \alpha\boldsymbol{p}_1) = f((3/2 - \alpha/2, 3/2 + \alpha/2)^{\mathrm{T}}) = \alpha^2/4 - (6/4)\alpha - 27/4$. $\phi'(\alpha) = 2\alpha/4 - 3/4 = 0 \Rightarrow \alpha = 3 > 1$. Hence, take $\alpha = 1$ and $\boldsymbol{x}_2 = (1,2)^{\mathrm{T}}$.

$\boldsymbol{x}_2 = (1,2)^{\mathrm{T}}$ is a KKT point. The objective function is convex (all eigenvalues to the Hessian are non-negative) and the feasible set is a polyhedron, so the problem is convex. The KKT conditions are sufficient for optimality for convex problems, so $\boldsymbol{x}_2 = (1,2)^{\mathrm{T}}$ is an optimal solution with $f(\boldsymbol{x}_2) = -8$.

Question 5

(Lagrangian duality)

(1p) a) The problem can be stated as that to minimize $f(\boldsymbol{x}) := \frac{1}{2}(x_1^2 + x_2^2)$ subject to the constraints that $x_1 + x_2 \geq 4$ and $x_j \leq 4$, $j = 1, 2$.

(1p) b) Introducing the Lagrange multiplier $\mu \geq 0$ for the constraint $x_1 + x_2 \geq 4$, the Lagrangian subproblem has the form

$$\underset{x_j \leq 4,\, j=1,2}{\text{minimize}} \; 4\mu + \frac{1}{2}x_1^2 - \mu x_1 + \frac{1}{2}x_2^2 - \mu x_2.$$

The problem separates over each variable, and the solutions are symmetric: for $0 \leq \mu \leq 4$, $x_j = \mu$ for $j = 1, 2$, while for $\mu > 4$, $x_j = 4$ for $j = 1, 2$. The explicit Lagrangian dual function hence is to maximize the function q over $\mu \geq 0$, where $q(\mu) = 4\mu - \mu^2$ for $0 \leq \mu \leq 4$, and $q(\mu) = 16 - 4\mu$ for $\mu \geq 4$. Its derivative hence is $q'(\mu) = 4 - 2\mu$ for $0 \leq \mu \leq 4$, and $q'(\mu) = -4$ for $\mu \geq 4$. The Lagrangian dual function clearly is concave over $\mu \geq 0$.

(1p) c) The solution to the Lagrangian dual problem is $\mu^* = 2$. Utilizing the result in b) we may derive that $\boldsymbol{x}^* = (2,2)^{\mathrm{T}}$. Strong duality holds, that is, $f(\boldsymbol{x}^*) = q(\mu^*)$.

(3p) Question 6

(optimality conditions)

See Theorem 10.10.

Question 7

(short questions)

(1p) a) X can be defined as an open set! Define $f(x) = x$ and $X = \{0 < x < 1\}$, the problem does not have an optimal solution.

(1p) b) The feasible set is convex (it is the line segment between (-1,0) and (1,0)). Th us KKT is sufficient (first question: yes). The set does not have an interior point, thus slater does not hold. LICQ does no t hold either. The objective $f(x, y) := x + y$ would result in an optimal solution at (-1,0), wh ich is not a KKT point, hence KKT is not necessary (second question: no).

(1p) c) We will use the notation $||a|| = \sqrt{\sum_{i=1}^{n} a_i^2}$. Assume that $||x - a||^2 \leq b$. We have that $||a - c|| = ||a - x + x - c|| \leq ||a - x|| + ||x - c||$, where the last inequality is the triangle inequality. Hence $||x - c|| \geq ||a - c|| - ||a - x|| \geq q \ln d + \sqrt{b} - \sqrt{b} = \ln d$. Therefore $\exp(||x - c||) \geq d$. This means that if we satisfy the first constr aint, then the other constraint is automatically satisfied (hence it is redundant). Since the first c onstraint is a convex function, the set is convex.
