Chalmers/Gothenburg University
Mathematical Sciences

EXAM SOLUTION

TMA947/MAN280
OPTIMIZATION, BASIC COURSE

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Question 1

(the simplex method)

(2p) a) We first rewrite the problem on standard form. We rewrite $x_1 = x_1^+ - x_1^-$ and introduce slack variables s_1 and s_2.

$$\begin{array}{lrcl}
\text{minimize} & x_1^+ - x_1^- - x_2 & & \\
\text{subject to} & x_1^+ - x_1^- + x_2 - s_1 & = & 1, \\
 & x_1^+ - x_1^- + 2x_2 + s_2 & = & 4, \\
 & x_1^+, x_1^-, x_2, s_1, s_2 & \geq & 0.
\end{array}$$

Phase I

We introduce an artificial variable a and formulate our Phase I problem.

$$\begin{array}{lrcl}
\text{minimize} & a & & \\
\text{subject to} & x_1^+ - x_1^- + x_2 - s_1 + a & = & 1, \\
 & x_1^+ - x_1^- + 2x_2 + s_2 & = & 4, \\
 & x_1^+, x_1^-, x_2, s_1, s_2 & \geq & 0.
\end{array}$$

We now have a starting basis (a, s_2). Calculating the reduced costs we obtain $\tilde{\mathbf{c}}_N = (-1, 1, -1, 1)^{\mathrm{T}}$, meaning that x_1^+ or x_2 should enter the basis. We choose x_2. From the minimum ratio rest, we get that a should leave the basis. This concludes phase I and we now have the basis (x_2, s_2).

Phase II

Calculating the reduced costs, we obtain $\tilde{\mathbf{c}}_N = (2, -2, 1)^{\mathrm{T}}$, meaning that x_1^- should enter the basis. From the minimum ratio test, we get that the outgoing variable is s_2. Updating the basis we now have (x_1^-, x_2) in the basis.

Calculating the reduced costs, we obtain $\tilde{\mathbf{c}}_N = (0, 3, 2)^{\mathrm{T}} \geq 0$, meaning that the current basis is optimal. The optimal solution is thus $(x_1^+, x_1^-, x_2, s_1, s_2)^{\mathrm{T}} = (0, 2, 3, 0, 0)^{\mathrm{T}}$, which in the original variables means $(x_1, x_2) = (-2, 3)^{\mathrm{T}}$, with optimal objective value $f^* = -5$.

(1p) b) The reduced costs are not all positive, so from the calculations we can not draw any conclusions regarding the uniqueness of the solution. However, the solution is unique in the original problem (draw the feasible set).

Question 2

(Lagrangian relaxation)

(2p) a) The dual problem is that to

$$\text{maximize}_{u\geq 0}\; q(u),$$

where q is the Lagrangian dual function defined as

$$\begin{aligned} q(u) &= \min_{0\leq x_j\leq 4, j=1,2} \Big((x_1-4)^2 + (x_2-2)^2 + u(x_1+x_2-4)\Big) && (1)\\ &= 20 - 4u + \min_{0\leq x_1\leq 4}\Big(x_1^2 - 8x_1 + ux_1\Big) + \min_{0\leq x_2\leq 4}\Big(x_2^2 - 4x_2 + ux_2\Big). && (2)\end{aligned}$$

The minimum of the two subproblems in (2) are attained at

$$x_1(u) = \frac{8-u}{2} \quad \text{and} \quad x_2(u) = \frac{4-u}{2}.$$

respectively. Inserting this into (1) we get that

$$q(u) = 2u - \frac{u^2}{2},$$

which attains is maximum when $q'(u) = 2 - u = 0$. So the optimal dual solution is $u^* = 2$ with dual objective value $q^* = q(u^*) = 2$.

(1p) b) At $u = 2$, we have that $x_1(u) = 3$ and $x_2(u) = 1$. This is a feasible solution to the primal problem with objective value 2, which is the same as the dual optimal value, implying that $\boldsymbol{x}^* = (3,1)^{\mathrm{T}}$ is an optimal solution to the primal problem.

Question 3

(algorithm choice)

(1p) a) The Frank–Wolfe method is most appropriate; exterior penalty is also applicable but makes less assumptions on problem structure, the others are not applicable. (Differentiable objective function, the feasible set is a bounded polyhedron).

(1p) b) The subgradient method is most appropriate. The only other candidate for an applicable methods is the exterior penalty method, but it does not use the convexity of the problem. Further, without checking it is unclear whether the Lagrangian function is differentiable or not. (Lagrangian dual problems are convex, and the subgradients can easily be computed).

(1p) c) Exterior penalty is the only applicable method.

Question 4

cones and conditions

(1p) a) $T_S(\boldsymbol{x}) = \{\boldsymbol{0}\}$ for all $\boldsymbol{x} \in S$, since for any sequence $\{\boldsymbol{x}_k\} \subset S$, $\boldsymbol{x}_k \to \boldsymbol{x}$ we must have $\boldsymbol{x}_k = \boldsymbol{x}$ for all $k \geq K$ for some K.

(2p) b) For any $\boldsymbol{x} \in S$, since $T_S(\boldsymbol{x}) = \{\boldsymbol{0}\}$ and (by assumption) Abadie's CQ holds we have $G(\boldsymbol{x}) = T_S(\boldsymbol{x}) = \{\boldsymbol{0}\}$. Thus $G(\boldsymbol{x}) \cap F_0(\boldsymbol{x}) = \emptyset$, so all points $\boldsymbol{x} \in S$ are KKT-points (this can also be verified directly from solving the KKT-system). Although all feasible points are KKT-points, none is optimal, as the objective function is unbounded from below as $x_1 \to -\infty$.

(3p) Question 5

modelling

We declare variables x_{ik} for $i \in \mathcal{N}$, $k \in \mathcal{B}_i$, to be understood as the amount of cash bet k of game i. Further declare the variables y_i as the worst case payout from game i for $i \in \mathcal{N}$. A model can then be written as

$$\begin{aligned}
\text{maximize} \quad & \sum_{i \in \mathcal{N}} y_i & , \qquad & (1)\\
\text{subject to} \quad & y_i \leq r_{ik} x_{ik}, & i \in \mathcal{N}, k \in \mathcal{B}_i, \qquad & (2)\\
& \sum_{i \in \mathcal{N}} \sum_{k \in \mathcal{B}_i} x_{ik} = M, & \qquad & (3)\\
& x_{ik} \geq 0, & i \in \mathcal{N}, k \in \mathcal{B}_i. \qquad & (4)
\end{aligned}$$

The objective function (1) maximizes the worst case scenario payout. The inequalities (2) models that the worst case scenario payout is less than the payout for any outcome. The equality (3) states that the total amount of bets to be made is M SEK. The final inequalities (4) are definitional and state that we cannot bet negative money.

(3p) Question 6

(strong duality in linear programming)

See Theorem 10.6 in The Book.

Question 7

(true or false)

(1p) a) False. If no feasible solution exists, the optimal value is > 0. If feasible solutions exist, the optimal value is $= 0$. (Section 9.1.2.)

(1p) b) True. We have that $\nabla f(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{d} = -\boldsymbol{d}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{d} < 0$, since $\boldsymbol{G}$ is a positive definite matrix (Section 2.2, page 37). Then, by Proposition 4.16, $\boldsymbol{d}$ is a descent direction for f at $\boldsymbol{x}$ since $\nabla f(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{d} < 0$. Hence (Definition 4.15) $\exists \delta > 0$ such that $f(\boldsymbol{x} + t\boldsymbol{d}) < f(\boldsymbol{x})$ for all $t \in (0, \delta]$.

(1p) c) False. Consider the function $g(x) = 4 - x^2$ and the two points $x^1 = -2$ and $x^2 = 3$ which belong to the set $S = \{\, x \in \mathbb{R} \mid g(x) \leq 0 \,\}$. By Definitions 3.31 and 3.32, g is concave. The point $\frac{1}{2} \cdot x^1 + \frac{1}{2} \cdot x^2 = \frac{1}{2} \notin S$. Hence, by Definition 3.1, the set S is not convex.
