Chalmers/GU
Mathematics

EXAM SOLUTION

TMA947/MMG621 NONLINEAR OPTIMISATION

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Question 1

(the simplex method)

(2p) a) We first rewrite the problem on standard form. We introduce slack variables s_1 and s_2. Consider the following linear program:

$$\begin{array}{llrrrrrl} \text{minimize} & z = x_1 - & x_2 + & x_3 & & & & \\ \text{subject to} & & 2x_2 + & x_3 + & s_1 & & = & 5, \\ & x_1 - & x_2 + & 2x_3 & & - s_2 & = & 5, \\ & x_1, & x_2, & x_3, & s_1, & s_2 & \geq & 0. \end{array}$$

An obvious starting basis is (s_1, x_1) and we can thus begin directly with *Phase II*. Calculating the reduced costs we obtain $\tilde{\mathbf{c}}_N = (0, -1, 1)^{\mathrm{T}}$, meaning that x_3 enters the basis. From the minimum ratio test, we get that x_1 leaves the basis.

Updating the basis we now have (s_1, x_3) in the basis. Calculating the reduced costs, we obtain $\tilde{\mathbf{c}}_N = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})^{\mathrm{T}}$. meaning that x_2 enters the basis. From the minimum ratio test, we get that the outgoing variable is s_1.

Updating the basis we now have (x_2, x_3) in the basis. Calculating the reduced costs, we obtain $\tilde{\mathbf{c}}_N = (\frac{2}{5}, \frac{1}{5}, \frac{3}{5})^{\mathrm{T}} \geq 0$, meaning that the current basis is optimal. The optimal solution is thus

$$(x_1, x_2, x_3, s_1, s_2)^{\mathrm{T}} = (0, 1, 3, 0, 0)^{\mathrm{T}},$$

which in the original variables means $(x_1, x_2, x_3)^{\mathrm{T}} = (0, 1, 3)^{\mathrm{T}}$ with optimal objective value $f^\star = 2$.

(1p) b) Calculating the reduced costs of the problem for the optimal basis of the problem from a), we obtain $\tilde{\mathbf{c}}_N = (\frac{3}{5} + \frac{1}{5}\alpha, -\frac{1}{5} - \frac{2}{5}\alpha, \frac{2}{5} - \frac{1}{5}\alpha)^{\mathrm{T}} \geq 0$ meaning that the the optimal solution from a) remains optimal for $-3 \leq \alpha \leq -\frac{1}{2}$.

(3p) Question 2

(convexity)

This is Theorem 4.3.

Question 3

(KKT optimality conditions)

(1p) a) With $\boldsymbol{z} = (2, 3/2)^{\mathrm{T}}$, the optimization problem to solve is that to

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{z}\|^2, \\ \text{subject to} \quad & x_1 + x_2 \leq 3/2, \\ & x_j \geq 0, \; j = 1, 2. \end{aligned}$$

The objective function is clearly a convex function and the feasible set is a convex set. Hence, the optimization problem is a convex optimization problem.

(1p) b) The KKT conditions for a feasible vector $\boldsymbol{x}^*$ are as follows:

$$\begin{aligned} \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} - \begin{pmatrix} 2 \\ 3/2 \end{pmatrix} + \mu_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \mu_3 \begin{pmatrix} 0 \\ -1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \mu_1(x_1^* + x_2^* - 3/2) &= 0, \\ \mu_2 x_1^* &= 0, \\ \mu_3 x_3^* &= 0, \\ \mu_j &\geq 0, \quad j = 1, 2, 3. \end{aligned}$$

All constraints are affine, so the KKT conditions are necessary for optimality. Since it is a convex optimization problem, the KKT conditions are also sufficient for optimality.

(1p) c) At $\boldsymbol{x}^* = (1, 1/2)^{\mathrm{T}}$, it must hold that $\mu_2 = 0$ and $\mu_3 = 0$. The remaining part of the KKT conditions is then:

$$\mu_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which has solution $\mu_1 = 1 \geq 0$. Hence, the point $\boldsymbol{x}^*$ is a KKT. From b) we know that it is also an optimal solution.

Question 4

(the gradient projection algorithm)

We have that $\nabla f(\boldsymbol{x}) = (2x_1 + x_2 - 10, x_1 + 4x_2 - 4)^{\mathrm{T}}$. So $\nabla f(\boldsymbol{x}_0) = (-5, 2)^{\mathrm{T}}$ and $\boldsymbol{x}_0 - \alpha_0 \nabla f(\boldsymbol{x}_0) = (9/2, 0)^{\mathrm{T}}$. Performing the projection we get that $\boldsymbol{x}_1 = \operatorname{Proj}_X(\boldsymbol{x}_0 - \alpha_0 \nabla f(\boldsymbol{x}_0)) = \operatorname{Proj}_X\left((9/2, 0)^{\mathrm{T}}\right) = (2, 0)^{\mathrm{T}}$.

It holds $\nabla f(\boldsymbol{x}_1) = (-6, -2)^{\mathrm{T}}$ and $\boldsymbol{x}_1 - \alpha_1 \nabla f(\boldsymbol{x}_1) = (7/2, 1/2)^{\mathrm{T}}$. Performing the projection we get that $\boldsymbol{x}_2 = \operatorname{Proj}_X(\boldsymbol{x}_1 - \alpha_1 \nabla f(\boldsymbol{x}_1)) = \operatorname{Proj}_X\left((7/2, 1/2)^{\mathrm{T}}\right) = (2, 1/2)^{\mathrm{T}}$.

The point $\boldsymbol{x}_2$ is actually a global minimum. This can be verified by either taking another step with the algorithm or by noting that the point is a KKT-point.

(3p) Question 5

(modelling)

For each word w_i, we introduce a binary decision variable x_i such that $x_i = 1$ if and only if word w_i is built. For each pair of words w_i and w_j with $j > i$, a binary variable y_{ij} is used. If $x_i = 0$ or $x_j = 0$ we require that $y_{ij} = 0$. A model can then be written as

$$\begin{aligned}
\text{maximize} \quad & \sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} b_{ij} y_{ij} \\
\text{subject to} \quad & \sum_{i=1}^{n} o_{i\alpha} x_i \leq N_\alpha, \quad && \forall \alpha \\
& y_{ij} \leq x_i, && i, j = 1, \dots, n, j > i \\
& y_{ij} \leq x_j, && i, j = 1, \dots, n, j > i \\
& y_{ij} \in \{0, 1\}, && i, j = 1, \dots, n, j > i \\
& x_i \in \{0, 1\}, && i = 1, \dots, n.
\end{aligned}$$

The program is linear with binary variables.

Question 6

(true or false)

(1p) a) False. Take $f(x) := e^x$.

(1p) b) False. Take $f(x) := x^4$ at $x = 0$.

(1p) c) True. See Proposition 3.65.

(3p) Question 7

(the Separation Theorem)

This is Theorem 4.28.
