Chalmers/GU
Mathematics

EXAM SOLUTION

TMA947/MMG621
NONLINEAR OPTIMISATION

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Question 1

(the simplex method)

(2p) a) We first rewrite the problem on standard form by introducing slack variables s_1 and s_2. Consider the following linear program:

$$\begin{array}{lrrrrl} \text{minimize} & -5x_1 & -4x_2 & & & \\ \text{subject to} & x_1 & & +s_1 & & =7, \\ & x_1 & -x_2 & & +s_2 & =8, \\ & x_1, & x_2, & s_1, & s_2 & \geq 0. \end{array}$$

The starting basis is $(s_1, s_2)^{\mathrm{T}}$. The reduced costs for the non-basic variables x_1 and x_2 are $\tilde{\mathbf{c}}_N = (-5, -4)^{\mathrm{T}}$, meaning that x_1 enters the basis. From the minimum ratio test, we get that s_1 leaves the basis.

Updating the basis we now have $(x_1, s_2)^{\mathrm{T}}$ in the basis. Calculating the reduced costs, we obtain $\tilde{\mathbf{c}}_N = (5, -4)^{\mathrm{T}}$, meaning that x_2 enters basis. From the minimum ratio test we get that $\mathbf{B}^{-1}\mathbf{N}_{x_2} = (0, -2)^{\mathrm{T}} \leq \mathbf{0}$, meaning that the problem is unbounded. The direction of unboundness is $\mathbf{p} = (x_1, x_2, s_1, s_2) = (0, 1, 0, 2)^{\mathrm{T}}$ and $z \to \infty$ along the half-line $l(\mu) = (7, 0, 0, 8)^{\mathrm{T}}$ $+\mu(0, 1, 0, 2)^{\mathrm{T}}$, $\mu \geq 0$.

(1p) b) For example $-x_1 + x_2 = 0$ can be added to get a uniquely solvable linear program. The optimal solution is then $\mathbf{x}^* = (7, 7, 0, 0)^T$ and $z^* = 63$.

(3p)
Question 2

(finiteness of the simplex algorithm)

Theorem 9.11 establishes the finite termination of the simplex method. The termination criterion is equivalent to the optimality conditions for the LP.

Question 3

(LP duality)

(1p) a) Since $q(\boldsymbol{\mu}) = \min_{i=\{1,\dots,N\}} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^i + \boldsymbol{\mu}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{x}^i - \boldsymbol{b})$, the dual (maximization) problem can be written as

$$\begin{aligned} \max_{\boldsymbol{\mu}} \quad & \min_{i\in\{1,\dots,N\}} \ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^i + \boldsymbol{\mu}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{x}^i - \boldsymbol{b}) \\ \text{subject to} \quad & \boldsymbol{\mu} \geq \boldsymbol{0} \end{aligned}$$

This is equivalent to (2) in the problem statement.

(1p) b) The LP dual of (2) in the problem statement is

$$\begin{aligned} \min_{\boldsymbol{\nu}} \quad & \sum_{i=1}^{N} \nu_i(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^i) \\ \text{subject to} \quad & \sum_{i=1}^{N} \nu_i(\boldsymbol{A}\boldsymbol{x}^i - \boldsymbol{b}) \leq 0 \\ & \sum_{i=1}^{N} \nu_i = 1, \ \ \boldsymbol{\nu} \geq \boldsymbol{0} \end{aligned}$$

This problem is equivalent to

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \in \operatorname{conv}\Big(\big\{\boldsymbol{x} \mid \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{d}, \ \boldsymbol{x} \in \{0,1\}^n\big\}\Big) \end{aligned} \tag{a}$$

because $\boldsymbol{x} \in \operatorname{conv}\Big(\big\{\boldsymbol{x} \mid \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{d}, \ \boldsymbol{x} \in \{0,1\}^n\big\}\Big)$ if and only if $\boldsymbol{x} = \sum_{i=1}^{N} \nu_i \boldsymbol{x}^i$ for some $\boldsymbol{\nu} \geq \boldsymbol{0}$, $\sum_{i=1}^{N} \nu_i = 1$. Problem (2) in the statement is feasible (e.g., $\boldsymbol{\mu} = \boldsymbol{0}$ and $y = \min_i \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^i$). In addition, the feasibility of (1) in the problem statement (i.e., the original integer program) implies that the dual of (2) in the problem statement is feasible. Hence, linear programming strong duality implies that the optimal objective values of (a) and (2) in the problem statement are the same.

(1p) c) If $\boldsymbol{x} \in \operatorname{conv}\Big(\big\{\boldsymbol{x} \mid \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{d}, \ \boldsymbol{x} \in \{0,1\}^n\big\}\Big)$ then $\boldsymbol{x}$ satisfies $\boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{d}$. Hence, the feasible set of (a) is included in the feasible set of the LP relaxation of (1) in the problem statement. Hence, $z^*_{LP} \leq z^*_{LD}$. Finally, the inequality $z^*_{LD} \leq z^*_{IP}$ is due to weak duality.

(3p) Question 4

(modelling)

The decision variables are:

b_i, $i = 1, \dots, 5$ width of segment i

h_i, $i = 1, \dots, 5$ height of segment i

Model

$$
\begin{aligned}
\text{minimize} \quad & l \sum_{i=1}^{5} b_i h_i, \\
\text{subject to} \quad & \frac{6Pl}{b_5 h_5^2} \leq \sigma_{\max}, \\
& \frac{6P(2l)}{b_4 h_4^2} \leq \sigma_{\max}, \\
& \frac{6P(3l)}{b_3 h_3^2} \leq \sigma_{\max}, \\
& \frac{6P(4l)}{b_2 h_2^2} \leq \sigma_{\max}, \\
& \frac{6P(5l)}{b_1 h_1^2} \leq \sigma_{\max}, \\
& \frac{Pl^3}{E}\left(\frac{244}{b_1 h_1^3} + \frac{148}{b_2 h_2^3} + \frac{76}{b_3 h_3^3} + \frac{28}{b_4 h_4^3} + \frac{4}{b_5 h_5^3}\right) \leq \delta_{\max}, \\
& \frac{h_i}{b_i} \leq a_{\max}, \; i = 1, \dots, 5, \\
& h_i \geq 0, b_i \geq 0, \; i = 1, \dots, 5.
\end{aligned}
$$

Question 5

(true or false)

(1p) a) False: at a stationary point the Hessian may have a negative eigenvalue, corresponding to an eigenvector $\boldsymbol{p}$, resulting in $\boldsymbol{p}^{\mathrm{T}} \nabla^2 f(\boldsymbol{x}) \boldsymbol{p} < 0$. This vector hence is a descent direction.

(1p) b) True: this is Theorem 6.4.

(1p) c) False: a global optimum is - by definition - also a local one.

(3p) Question 6

(optimality conditions)

The feasible set is nonempty and convex (three-dimensional box), z is C^1 on the feasible set and convex since its hessian $\mathbf{P}$ is positive semidefinite ($\mathbf{P}$ is symmetric and the upper left 1-by-1 corner of $\mathbf{P}$, 2-by-2 corner of $\mathbf{P}$ and $\mathbf{P}$ itself have positive determinants. Then Sylvester's criteria establishes the positive definiteness of $\mathbf{P}$. Eigenvalues of $\mathbf{P}$ can be found approximately, e.g. by bisection, instead to establish the positive definiteness of $\mathbf{P}$.).

Now we need to verify variational inequality to establish the global optimality of $\mathbf{x}^*$. The gradient of the objective function at $\boldsymbol{x}^*$ is

$$\nabla z(\boldsymbol{x}^*) = (-1, 0, 2)^{\mathrm{T}}.$$

Therefore the variational inequality is that

$$\nabla z(\boldsymbol{x}^*)^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x}) = -1(y_1 - 1) + 2(y_3 + 1) \geq 0$$

for all $\boldsymbol{y}$ satisfying $-1 \geq y_i \geq 1$, which is clearly true. So $\mathbf{x}^*$ is a global optimum of the problem considered (Theorem 4.23).

(3p) Question 7

(the basis of the SQP algorithm)

See equation (13.25) in the course book: the subproblem is equivalent to a second-order approximation of the KKT conditions of the original problem.
