TMA947 / MMG621 — Nonlinear optimization

Exercise 1 – Modelling and Convexity

October 23, 2017

E1.1 (easy) To produce a g. of cookies it is necessary to have b_1 g. of sugar and b_2 g. of wheat. Farmer A sells wheat at the price c_{11} and sugar at the price c_{12} and Farmer B sells wheat at the price c_{21} and sugar at the price c_{22}. Formulate a linear model that minimizes the cost of purchasing sugar and wheat for the production of d g. of cookies. Clearly state you variables, objective and constraints.

E1.2 (easy)

(a) Consider the circuit in Figure 1. You wish to decide the voltage V and the resistance $R1$ and $R2$ such that the power used at the power source is minimized, at least a power of P_1 is generated in $R1$ and P_2 in R_2, and the current I is at most $I_{\max}$. State you variables, objective and constraints. (Hint: The power is given by $P = VI$ and Ohm's law is $V = RI$)

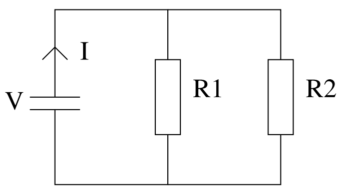

Figure 1: (E1.2)

(b) Assume that you only have resistors available with the resistance $r_1, r_2, \ldots, r_m$. Modify your model to accommodate this. (Hint: you must introduce binary variables).

E1.3 (easy) A company wishes to transport a tons of goods between Gothenburg (G) and Shanghai (S). There are no direct connections, but it is possible to make transfer connections according to Figure 2. Transporting goods between city i and j costs c_{ij} \$/ton and there is a maximum capacity of d_{ij} tons. Formulate an LP to minimize the transportation costs. State your variables, objective and constraints.

Figure 2: (E1.3)

E1.4 (easy) You are responsible for buying raw plastic in a toy production company during one week. The prices for buying plastic and the demand of plastic in the production vary during the week according to Table 1. It is possible to store plastic from one day to the next at the price of 500 SEK/ton. Construct a linear programming model for the optimal purchase of raw plastic during the week.

	cost [SEK/ton]	demand [ton]
Monday	1000	100
Tuesday	1500	200
Wednesday	2000	150
Thursday	1500	250
Friday	2500	300

Table 1: (E1.4)

E1.5 (easy) At one Saturday five games take place in the Swedish football league allsvenskan at five different locations. Each game has to be assigned one main referee and there are five referees available. The traveling cost of the referees to the different cities however vary according to Table 2. Construct an integer programming model which assigns a referee to each game such that the total traveling cost is minimized.

	Anders	Stefan	Johan	Lennart	Bengt
Göteborg	50	60	400	430	280
Stockholm	500	510	80	60	320
Malmö	600	550	500	300	250
Helsingborg	500	450	400	400	200
Kalmar	300	310	370	320	250

Table 2: (E1.5) The table lists the traveling cost in SEK to each game location for each referee.

E1.6 (medium) A chocolate producer wants to plan the yearly production of chocolate. He has to fulfill the demand for chocolate in each month according to Table 3. In order to produce 1 kg of chocolate he needs 0.7 kg cocoa and 0.3 kg sugar. He has the possibility to sign a deal with an importer for monthly deliveries of sugar and cocoa, the importer will then deliver the same amount of sugar and cocoa each month (the amount is decided by the chocolate producer, but has to be equal for all months). He can also buy the goods for a higher price at the local market, but has then the possibility to buy different amounts each month. The prices are presented in Table 4. If there are goods left after a month's production, they can be stored until the next month. There is however a maximal storage capacity of 100 kg.
Introduce appropriate constants and variables, and create a Linear Programming model that minimizes the yearly production costs.

jan	feb	mar	apr	may	jun	jul	aug	sep	okt	nov	dec
300	230	270	500	150	170	140	230	300	270	350	700

Table 3: (E1.6) The table displays the demand for chocolate in kg for each month.

	import	market
cocoa	50	70
sugar	10	12

Table 4: (E1.6) The table displays the prices of goods in kr/kg from import and local market.

E1.7 (hard) The government has assigned you to lead their aid program. They are willing to spend 1 % of the national gross product of b SEK. They are considering to give aid to a set of countries $\mathcal{N} = \{1, \dots, N\}$. The aim of the aid program is to increase the Human Development Index (HDI) in the countries which is calculated by measuring the three factors education per capita, life expectancy and gross national product per capita (GDP). We may therefore consider the HDI index as a measure of *development per capita* in a country.
For all countries $j \in \mathcal{N}$ let a_j denote the current value of the HDI index, c_j the increase of HDI per SEK given as aid to the country and p_j the population size. There are only a limited number of aid programs in each of the countries. This puts a limit on the maximal aid that a country can receive, let d_j SEK denote the maximal aid country j can receive.

(a) Write a linear program for distributing aid that maximizes the HDI in the region formed by all the countries considered.

(b) You are given new directives from the government, they think that the aid should be focused on a maximal number of M countries. Introduce integer variables in your model (i.e. write an Linear Integer Programming model) in order to accommodate this demand.

(c) There has recently been some discussion concerning aid to countries with high HDI. The government wants you to write a new model that maximizes the minimal HDI among the countries. Extend your model in a) to an LP model that accommodates this demand.

E1.8 (hard) You are in charge of delivering coffee to $\mathcal{M} = \{1, \dots, m\}$ cafes from $\mathcal{N} = \{1, \dots, n\}$ depots. The demand for coffee in cafe j is d_j. The cost of delivering x kg of Giffy from depot i to cafe j is given by the piecewise linear function $f_{ij}(x) := c_{ij}^k x + m_{ij}^k$ if $a_{ij}^k \leq x \leq a_{ij}^{k+1}$ for some $k = 1, \dots, K-1$. We have that $a_{ij}^1 = 0$ and $a_{ij}^K = d_j$. We only need to satisfy the demand in l out of the m cafes (the remaining cafes can obtain their coffee in a different way). Construct a linear integer model for the delivery of coffee at minimal cost.

E1.9 (hard) Consider a set of cities indexed $\mathcal{N} = \{1, 2, \dots, n\}$. The cost of traveling between city i and j is c_{ij}. Assume that city 1 is the home town of a traveling salesman and that he has to visit all the cities $2, \dots, n$. The traveling salesman problem (TSP) is to find a minimal cost route starting from city 1 and ending in city 1, such that every city is visited once. Figure 3 illustrates a TSP tour with five cities.

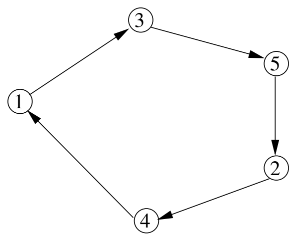

Figure 3: (E1.9) An example of a TSP tour with 5 cities. Here $x_{13} = 1$, $x_{35} = 1$, $x_{52} = 1$, $x_{24} = 1$ and $x_{41} = 1$ all other variables $x_{ij} = 0$.

Consider the following MIP model for the TSP. The variables are defined as

$$x_{ij} = \begin{cases} 1 & \text{travel to city j after city i,} \\ 0 & \text{otherwise.} \end{cases}$$

The model is then as follows.

$$\begin{aligned}
\text{minimize} \quad & \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} c_{ij} x_{ij} && \text{(1a)} \\
\text{subject to} \quad & \sum_{i \in \mathcal{N}} x_{ji} = 1, \qquad j \in \mathcal{N} && \text{(1b)} \\
& \sum_{i \in \mathcal{N}} x_{ij} = 1, \qquad j \in \mathcal{N} && \text{(1c)} \\
& x_{ii} = 0 \qquad i \in \mathcal{N}, && \text{(1d)} \\
& x_{ij} \in \{0, 1\}, \, i \in \mathcal{N}, \; j \in \mathcal{N}. && \text{(1e)}
\end{aligned}$$

(a) Present a solution to a problem with five cities illustrated in Figure 3 which satisfies the constraints in model (1), but is not an acceptable TSP route (Draw!).

(b) (a) shows that the constraints of model (1) are not enough to model the TSP. Introduce an additional linear constraint for the five city problem in Figure 3 such that the suggested solution in (a) becomes infeasible. (HINT: Consider constraint of the form $\sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{N} \setminus \mathcal{A}} x_{ij} \geq 1$. Specify the set $\mathcal{A} \subset \mathcal{N}$. Draw!)

(c) Generalize the procedure in (b) and extend model (1) with an additional set of constraints which assure that we always obtain a feasible route.

(d) The number of constraints of the model in (c) is large. Solving a model with such a large number of constraints will be possible only for a very small number of cities. Devise a more practical solution approach for TSP using the modelling in (c).

E1.10 (medium)

(a) Show that the set $S = \{x \in \mathbb{R}^3 | x_1 - x_2^2 \le x_3 \le x_1 + x_2^2\}$ is not convex.

(b) Let C be a nonempty subset of $\mathbb{R}^n$, and let λ_1 and λ_2 be positive scalars. Show that if C is convex, then $(\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C$. Use the following definitions:

- For nonempty $C \subseteq \mathbb{R}^n$ and $\lambda > 0$,
  $\lambda C = \{\lambda x \,|\, x \in C\}$
- For nonempty $C \subseteq \mathbb{R}^n$ and nonempty $D \subseteq \mathbb{R}^n$,
  $C + D = \{x + y \,|\, x \in C, y \in D\}$

Show by example that this need not be true when C is not convex.

(c) Find extreme points of the polyhedra defined below by algebraic and graphical methods.

$$\begin{cases} x_1 + 2x_2 \ge 2, \\ x_2 - x_1 \le 1, \\ x_1 \le 2, \\ x_2 \ge 0. \end{cases}$$

E1.11 (medium) Decide if the functions f defined below are convex or not.

(a)

$$f(\mathbf{x}) = \|\mathbf{x}\|^n \text{ with } n \ge 1.$$

(b)

$$f(\mathbf{x}) = a_1 x_1 + \cdots + a_n x_n.$$

(c)

$$f(\mathbf{x}) = pg(\mathbf{x}) - c(\mathbf{x}),$$

where $p > 0$, $g(\mathbf{x})$ concave, $c(\mathbf{x})$ convex.