Chalmers/GU
Mathematics

EXAM

TMA947/MMG621
NONLINEAR OPTIMISATION

Date:	17–08–24
Time:	8^{30}–13^{30}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Caroline Granfeldt, tel. 5325
Result announced:	17–09–14 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(simplex method)

The following linear optimization problem is given:

$$\begin{array}{llrrrl} \text{maximize} & z = & 4x_1 & + 2x_2 & + 2x_3, & \\ \text{subject to} & & x_1 & - \ x_2 & + 2x_3 & \leq 2, \\ & & 2x_1 & + \ x_2 & + \ x_3 & \leq 8, \\ & & x_1, & x_2, & x_3 & \geq 0. \end{array}$$

(2p) a) Solve the problem using phase I and phase II of the simplex method.

Aid: You may utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(1p) b) State the LP dual to the above problem and solve it graphically. Does it have an optimal solution?

(3p) Question 2

(Lagrangian duality)

For a symmetric real matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ consider the problem to find

$$\text{minimum}_x \, f(\boldsymbol{x}) := -\boldsymbol{x}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{x}, \tag{1}$$

subject to the constraint that $\boldsymbol{x}^{\mathrm{T}} \boldsymbol{x} = 1$.

Derive the KKT conditions to this problem, and interpret the solution.

(3p) Question 3

(Newton's method)

An engineer has decided to verify numerically that the exponential function $x \to \exp(x) = e^x$ grows faster than any polynomial. In order to do so, he/she studies the optimization problem to

$$\underset{x \in \mathbb{R}}{\text{minimize}}\ f(x) = x^\alpha - e^x, \tag{1}$$

where α is the highest power of the polynomial (we assume it is an even, positive integer number). The engineer uses a Newton method (with unit steps!) to solve the problem. He/she argues that if the exponential function grows faster than any polynomial, then the sequence $\{x^k\}$ generated by the method should diverge to infinity, because the objective function f can be decreased indefinitely by increasing the value of x.

(1p) a) State the Newton iteration explicitly for the given problem (1).

(1p) b) Construct a numerical example (that is, choose a value of $\alpha \in \{2, 4, ...\}$ and a starting point of the Newton algorithm) illustrating the engineers error in reasoning.

(1p) c) Find the error in the engineer's reasoning and formally explain it.

(3p) Question 4

(unconstrained optimization)

Suppose that you have attacked the problem of minimizing a differentiable function f over $\mathbb{R}^n$. Explain as well as you can how you can measure, and motivate, whether or not a vector $\boldsymbol{x}$ is near-stationary.

(3p) ## Question 5

(modelling)

Suppose that people from two groups, G_1 and G_2, wish to pair up with each other. Each group contains $n \in N$ people, and thus the total number of pairings will be n as well. A pair has to consist of one person from each group.

All persons have been asked to rank people from the other group with numbers, where the higher number means a higher preferability to be paired up. For person $i \in G_1$, the ranking of person $j \in G_2$ is a_{ij}, and for person $j \in G_2$, the ranking of person $i \in G_1$ is b_{ji}. Thus, if person $i \in G_1$ and person $j \in G_2$ pair up, the sum of the rankings of these two persons is $a_{ij} + b_{ji}$. If, however, the individual ranking is a value below p, it means a person really don't like the other one and therefore refuses to be paired up with that person. In other words, these two persons can't pair up.

Formulate an integer linear model which pairs up people from both groups while maximizing the sum of the total rankings.

Question 6

(true or false)

The below three individual claims should be assessed individually. Are they true or false, or is it impossible to say? For each of the three statements, provide an answer, together with a short—but complete— motivation.

(1p) a) *Claim:* A convex quadratic function always has a minimum over $\mathbb{R}^n$.

(1p) b) *Claim:* A linear program always has a non-empty polyhedral set of optimal solutions.

(1p) c) *Claim:* The Lagrangian dual problem to any problem is a convex one.

Question 7

(a simple optimization problem)

In a recent optimization exam at a Swedish technical university, the following optimization problem was addressed:

$$
\begin{aligned}
\text{maximize} \quad & f(\boldsymbol{x}) := \sum_{j=1}^{n} a_j/x_j, \\
\text{subject to} \quad & \sum_{j=1}^{n} \log x_j \leq b, \\
& x_j > 0, \qquad j = 1, \dots, n,
\end{aligned}
$$

where $a_j > 0$ for all j, and $b > 0$.

The students where asked to derive the optimal solution to this problem through a Lagrangian relaxation of the first constraint, and by then solving the resulting dual problem. Explain what is wrong with this exam question. In other words, prove that there does not exist an optimal solution to this problem.

[Hint: Utilize the KKT conditions.]