Chalmers/GU
Mathematics

EXAM

TMA947/MMG621
NONLINEAR OPTIMISATION

Date:	18–08–21
Time:	8^{30}–13^{30}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Michael Patriksson, tel. 709-581812
Result announced:	18–09–09 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$
\begin{aligned}
\text{maximize} \quad & z = 3x_1 + 2x_2, \\
\text{subject to} \quad & 2x_1 + 3x_2 \leq 1, \\
& x_1 - x_2 \geq 4, \\
& x_1, \quad x_2 \geq 0.
\end{aligned}
$$

(2p) a) Solve the problem using phase I and phase II of the simplex method.

Aid: You may utilize the identity

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
$$

(1p) b) If an optimal solution exists, then use your calculations to decide whether it is unique or not. If the problem is unbounded, then use your calculations to specify a direction of unboundedness of the objective value.

Question 2

Suppose we have an LP problem (the primal one)

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & Ax \geq b, \\ & x \geq 0^n. \end{aligned}$$

(1p) State the LP dual problem.

(2p) Suppose that this dual problem is feasible. Prove, or disprove, whether the primal problem has an optimal solution, or not. Motivate clearly.

Question 3

(feasible direction methods)

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := (x_1 - 1/2)^2 + x_2^2, \\ \text{subject to} \quad & 0 \leq x_1 \leq 1, \\ & 0 \leq x_2 \leq 1. \end{aligned}$$

(2p) Draw the first two iterates obtained using the Frank-Wolfe algorithm starting in $\boldsymbol{x}_0 = (0, 1)^{\mathrm{T}}$.

(1p) Draw the first two iterates obtained using the simplicial decomposition algorithm starting in $\boldsymbol{x}_0 = (0, 1)^{\mathrm{T}}$.

(3p) Question 4

(on the SQP algorithm and the KKT conditions)

Consider the following nonlinear programming problem: find $\boldsymbol{x}^* \in \mathbb{R}^n$ that solves the problem to

$$\begin{aligned} \text{minimize} \;\; & f(\boldsymbol{x}), && && \text{(1a)} \\ \text{subject to} \;\; & g_i(\boldsymbol{x}) \leq 0, && i = 1, \dots, m, && \text{(1b)} \\ & h_j(\boldsymbol{x}) = 0, && j = 1, \dots, \ell, && \text{(1c)} \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, g_i, and $h_j : \mathbb{R}^n \to \mathbb{R}$ are given functions in C^1 on $\mathbb{R}^n$.

We are interested in establishing that the classic SQP subproblem tells us whether an iterate $\boldsymbol{x}_k \in \mathbb{R}^n$ satisfies the KKT conditions or not, thereby establishing a natural termination criterion for the SQP algorithm.

Given the iterate $\boldsymbol{x}_k$, the SQP subproblem is to

$$\begin{aligned} \underset{p}{\text{minimize}} \;\; & \frac{1}{2}\boldsymbol{p}^{\mathrm{T}}\boldsymbol{B}_k\boldsymbol{p} + \nabla f(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p}, && && \text{(2a)} \\ \text{subject to} \;\; & g_i(\boldsymbol{x}_k) + \nabla g_i(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p} \leq 0, && i = 1, \dots, m, && \text{(2b)} \\ & h_j(\boldsymbol{x}_k) + \nabla h_j(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p} = 0, && j = 1, \dots, \ell, && \text{(2c)} \end{aligned}$$

where the matrix $\boldsymbol{B}_k \in \mathbb{R}^{n \times n}$ is symmetric and positive semi-definite.

Establish the following statement: the vector $\boldsymbol{x}_k$ is a KKT point in the problem (1) if and only if $\boldsymbol{p} = \boldsymbol{0}^n$ is a globally optimal solution to the SQP subproblem (2). In other words, the SQP algorithm terminates if and only if $\boldsymbol{x}_k$ is a KKT point.

Hint: Compare the KKT conditions of (1) and (2).

(3p) Question 5

(modelling)

A small municipality is forced to close one or several schools. Out of ten existing schools, at most three schools can be closed. The annual cost to keep school i open is c_i kr, where $i = 1, 2, \ldots, 10$. School i can educate a maximum of k_i students.

The municipality is divided into J home areas and there is a requirement that all students in an area belong to the same school. There are b_j students in area j and the average distance from area j to school i is d_{ij} km. The estimated annual cost for student travels is set to m kr per km and student.

Formulate a *linear integer program* to decide on which schools to keep and which ones to close, such that we minimize the total cost for schools and travels and fulfill the above listed requirement

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Question 6

(true or false)

Indicate for each of the following three statements whether it is true or false. Motivate your answers!

(1p) a) For the phase I (when a BFS is *not* known a priori) problem of the simplex algorithm, the optimal value is always zero.

(1p) b) Suppose that the function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is continuously differentiable on $\mathbb{R}^n$ and let $\boldsymbol{G}$ be a symmetric and positive definite matrix of dimension $n \times n$. Then, if $\nabla f(\boldsymbol{x}) \neq \mathbf{0}^n$ and the vector $\boldsymbol{d}$ fulfils $\boldsymbol{Gd} = -\nabla f(\boldsymbol{x})$ it holds that $f(\boldsymbol{x} + t\boldsymbol{d}) < f(\boldsymbol{x})$ for small enough values of $t > 0$.

(1p) c) If the function $g : \mathbb{R}^n \mapsto \mathbb{R}$ is concave on $\mathbb{R}^n$ and $c \in \mathbb{R}$, then the set $\{\, \boldsymbol{x} \in \mathbb{R}^n \mid g(\boldsymbol{x}) \leq c \,\}$ is convex.

(3p) Question 7

(the gradient projection algorithm)

The gradient projection algorithm is a generalization of the steepest descent method to problems over convex sets. Given a feasible point $\boldsymbol{x}_k$, the next point is obtained according to $\boldsymbol{x}_{k+1} = \text{Proj}_X(\boldsymbol{x}_k - \alpha_k \nabla f(\boldsymbol{x}_k))$, where X is the convex set over which we minimize, $\alpha_k > 0$ is the step length, and $\text{Proj}_X(\boldsymbol{y}) = \arg\min_{\boldsymbol{x} \in X} ||\boldsymbol{x} - \boldsymbol{y}||$ denotes the closest point to $\boldsymbol{x}$ in X.

Consider the problem to

$$
\begin{aligned}
\text{minimize} \quad & f(\boldsymbol{x}) := x_1^2 + 2x_2^2 - 2x_1x_2 - 2x_1 - 3x_2, \\
\text{subject to} \quad & 0 \leq x_1 \leq 3, \\
& 0 \leq x_2 \leq 2.
\end{aligned}
$$

Start at the point $\boldsymbol{x}_0 = (0, 0)^{\mathrm{T}}$ and perform two iterations of the gradient projection algorithm using step length $\alpha_k = 1$ for all k. You may solve the projection problem in the algorithm graphically. Is the point obtained a global/local minimum? Motivate why/why not.