Chalmers/GU
Mathematics

EXAM

TMA947/MMG621
NONLINEAR OPTIMISATION

Date:	18–11–01
Time:	14^{00}–19^{00}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Gustav Kettil, tel. 5325
Result announced:	18–11–21
	Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$
\begin{aligned}
\text{maximize} \quad & z = x_1 + 4x_2, \\
\text{subject to} \quad & x_1 + 3x_2 \leq 8, \\
& 2x_1 + x_2 \geq 4, \\
& x_2 \geq 0.
\end{aligned}
$$

(2p) a) Solve the problem using phase I and phase II of the simplex method.

Aid: You may utilize the identity

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
$$

(1p) b) If an optimal solution exists, then use your calculations to decide whether it is unique or not. If the problem is unbounded, then use your calculations to specify a direction of unboundedness of the objective value.

(3p) **Question 2**

(necessary local and sufficient global optimality conditions)

Consider and optimization problem of the following general form:

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}), && \text{(1a)} \\ \text{subject to} \quad & \boldsymbol{x} \in S, && \text{(1b)} \end{aligned}$$

where $S \subseteq \mathbb{R}^n$ is nonempty, closed and convex, and $f : \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}$ is in C^1 on S.

Prove the following two propositions.

PROPOSITION 1. *If $\boldsymbol{x}^* \in S$ is a local minimum of f over S then it hold that*

$$\nabla f(\boldsymbol{x}^*)^T(\boldsymbol{x} - \boldsymbol{x}^*) \geq 0, \qquad \boldsymbol{x} \in S. \tag{2}$$

PROPOSITION 2. *Suppose that f is convex on S. Then*

$$\boldsymbol{x}^* \textit{is a global minimum of } f \textit{ over } S \qquad \Longleftrightarrow \qquad (2) \textit{ holds.}$$

Question 3

(Unconstrained optimization)

Consider the following optimization problem where the objective is to

$$\begin{aligned} \text{minimize} \quad & x_1^2 + 6x_1x_2 + x_2^2, \\ \text{subject to} \quad & \boldsymbol{x} \in \mathbb{R}^2. \end{aligned}$$

Let $\boldsymbol{x}_0 = (7, 7)^{\mathrm{T}}$

(2p) a) Start at $\boldsymbol{x}_0$ and perform one step of Levenberg–Marquardt method with an Armijo line search, using the multiplier $\gamma = 6$, the fraction requirement $\mu = 0.8$.

(1p) b) For this problem, will Levenberg–Marquardt method converges to a global optimal or not?

Question 4

(KKT conditions)

For the set described by the system (1) Abadie's CQ holds, but what about the following constraint qualifications?

$$S := \left\{ x_1, x_2 \in \mathbb{R} \;\middle|\; \begin{array}{ll} (x_1-1)^2 + x_2^2 & \leq 1 \\ (x_1+3)^2 + (x_2-4)^2 & \geq 25 \\ (x_1+3)^2 + (x_2+4)^2 & \geq 25 \\ x_2 & \leq 2 \end{array} \right\} \tag{1}$$

(1p) a) Show whether the affine CQ is satisfied or not.

(1p) b) Show whether the Slater CQ is satisfied or not.

(1p) c) Show whether the LICQ is satisfied or not at the point $\bar{\boldsymbol{x}}^{\mathrm{T}} = (0, 0)$.

(3p) Question 5

(modelling)

IT-support have grown tired of purchasing computer hardware with employee–specific performance requirements, and wishes to be aided in the decision. A computer consists of six essential components: motherboard (MB), power supply unit (PSU), CPU, GPU, RAM, and harddrive (HD). There exist several models (variants) of each component and the problem is to choose a variant for each component. For two of the components the choice of variant needs to be taken with extra care. First, each variant of the MB has a specific interface, making it incompatible with some variants of other components. Thus, a variant of MB cannot be used with some variants of other components. Second, the PSU distributes power to all other components of the computer. Hence, the power capacity of the PSU must exceed the total power usage of all other components. Both cost and power usage of each variant of a component, are known parameters. Only some variants of each component are considered to meet the performance requirements. Thus, only such acceptable variants are used in the optimization problem.

State an integer linear model that minimizes the hardware cost for a single computer by deciding the variant of each component.

Question 6

(true or false)

Indicate for each of the following three statements whether it is true or false. Motivate your answers!

(1p) a) In the interior penalty method we let the penalty tend to infinity.

(1p) b) The set $S := \{\, x_1, x_2 \in \mathbb{R} \mid x_2 \sin x_1 \geq 0, x_2^2 + x_1^3 \leq 27, x_2 \geq 0, x_1 \geq -3 \,\}$ is convex.

(1p) c) There exists a convex problem on the form $\min_{x \in \mathbb{R}^4} f(\boldsymbol{x})$ subject to $g_i(\boldsymbol{x}) \leq 0$, $i = 1, \ldots, 3$, with an optimal primal solution $\boldsymbol{x}^*$, and Lagrangian multiplier vector $\boldsymbol{\mu}^*$, such that

$$\boldsymbol{x}^* = \begin{pmatrix} 3 \\ 4 \\ 7 \\ 5 \end{pmatrix}, \qquad \boldsymbol{\mu}^* = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{g}(\boldsymbol{x}^*) = \begin{pmatrix} -1 \\ -2 \\ -8 \end{pmatrix}.$$

(3p) Question 7

(linear programming duality)

Consider the following two polyhedral sets corresponding to the feasible sets of a pair of primal-dual linear programs:

$$X = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0} \,\},$$
$$Y = \{\, \boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}, \quad \boldsymbol{y} \geq \boldsymbol{0} \,\}.$$

Prove that if both sets X and Y are non-empty, then at least one of them must be unbounded.
