Lecture 8

Linear programming (I) - intro & geometry

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Linear programs (LP)

Consider a linear program (LP):

$$\begin{aligned} z^* = \text{infimum} \quad & \boldsymbol{c}^T \boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{x} \in P, \end{aligned}$$

where P is a polyhedron (i.e., $P = \{x \mid Ax \leq b\}$).

- $A \in \mathbb{R}^{m \times n}$ is a given matrix, and b is a given vector,
- Inequalities interpreted entry-wise (i.e., $(Ax)_i \leq (b)_i, \; i = 1, \dots, m$),
- Minimize a linear function, over a polyhedron (i.e., solution set of finitely many linear inequality constraints).

Polyhedra in different forms

Inequality constraints $Ax \leq b$ (i.e., $x \in P$) might look restrictive, but in fact more general:

- $x \geq \mathbf{0}^n \iff -I^n \boldsymbol{x} \leq \mathbf{0}^n$,
- $Ax \geq b \iff -Ax \leq -b$,
- $Ax = b \iff Ax \leq b \quad \text{and} \quad -Ax \leq -b$.

We often consider **polyhedron in standard form**:

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{Ax} = \boldsymbol{b},\ \boldsymbol{x} \geq \mathbf{0}^n\}.$$

P is a polyhedron, since $P = \{\boldsymbol{x} \in \mathbb{R}^n \mid \tilde{A}\boldsymbol{x} \leq \tilde{b}\}$ for some $\tilde{A}$ and $\tilde{b}$.

Standard form linear programs

We say that a LP is written in **standard form** if

$$\begin{aligned} z^* = \text{infimum} \quad & \boldsymbol{c}^T \boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

- Meaning that $P = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}$.
- Without loss of generality, we can assume $b \geq \boldsymbol{0}$.
- Standard form LP can in fact model all LP's.

Rewriting to standard form LP

- For example, we can add **slack variables** to transform inequality form LP into standard form LP.

$$(I): \quad \begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^T\boldsymbol{x}, \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \\ & \boldsymbol{x} \geq \mathbf{0}. \end{array} \qquad (II): \quad \begin{array}{ll} \underset{\boldsymbol{x},s}{\text{minimize}} & \boldsymbol{c}^T\boldsymbol{x}, \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} + \mathbf{s} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \mathbf{0}, \quad \mathbf{s} \geq \mathbf{0}. \end{array}$$

$\boldsymbol{x}^\star$ optimal to (I) $\iff$ $(\boldsymbol{x}^\star, s^\star)$ optimal to (II) for some $s^\star \geq \mathbf{0}$.

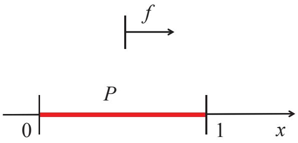
- If some variable x_j is not sign-constrained, substitute by

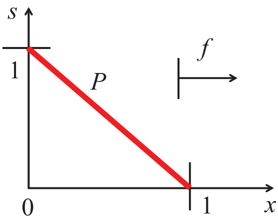
$$x_j = x_j^+ - x_j^-, \quad x_j^+, x_j^- \geq 0$$

- Equivalent linear programs do not need to have same feasible set.

Rewriting to standard form, example

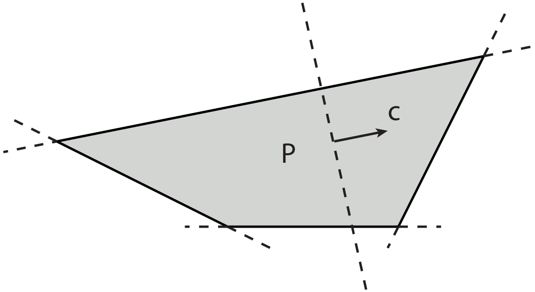
$$\begin{array}{ll} \text{minimize} & -2x \\ \text{subject to} & x \leq 1 \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} \text{minimize} & -2x \\ \text{subject to} & x + \mathbf{s} = 1 \\ & x, s \geq 0 \end{array}$$



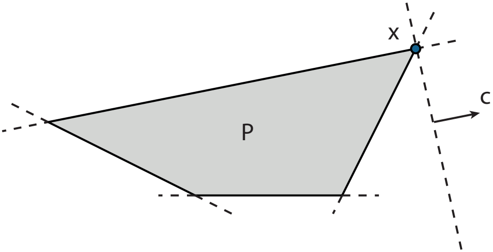


Equivalent linear programs, but different polyhedra!

$$
\begin{aligned}
z^* = \text{infimum} \quad & \boldsymbol{c}^T \boldsymbol{x}, \\
\text{subject to} \quad & \boldsymbol{x} \in P,
\end{aligned}
$$



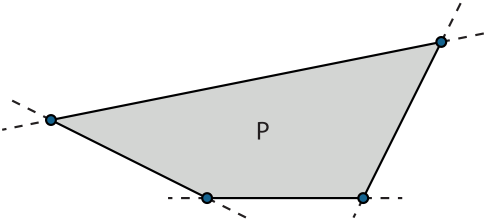
$$\begin{aligned} z^* = \text{infimum} \quad & \boldsymbol{c}^T \boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{x} \in P, \end{aligned}$$



- Optimality attained at extreme point.

Extreme point

An **extreme point** of a convex set S is a point that cannot be written as a convex combination of two other points in S.



- Extreme point has algebraic equivalence: basic feasible solution (BFS).

Basic solution (I)

Standard form polyhedron $P = \{x \mid Ax = b, x \geq \mathbf{0}\}$, $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) = m$

A point $\bar{\boldsymbol{x}}$ is a **basic solution** if

- $\boldsymbol{A}\bar{\boldsymbol{x}} = \boldsymbol{b}$, and
- the columns of $\boldsymbol{A}$ corresponding to non-zero components of $\bar{\boldsymbol{x}}$ are linearly independent

  (Recall that: $\boldsymbol{A}\bar{\boldsymbol{x}} = \sum_{j=1}^{n} \boldsymbol{a}_j \bar{x}_j$, where $\boldsymbol{a}_j$ is column j of $\boldsymbol{A}$.)

Basic solution (II)

Standard form polyhedron $P = \{x \mid Ax = b, x \geq \mathbf{0}\}$, $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) = m$

Procedure for constructing basic solution $\bar{\boldsymbol{x}}$

1. Choose m linearly independent columns of A
2. Rearrange $A = (B, N)$, with $B \in \mathbb{R}^{m \times m}$ and $\text{rank}(B) = m$, so

   $$Ax = Bx_B + Nx_N = b, \quad x_B = (\bar{x}_1, \ldots, \bar{x}_m),\ x_N = (\bar{x}_{m+1}, \ldots, \bar{x}_n)$$

3. Set $x_N = \mathbf{0}^{n-m}$, denoted **nonbasic variables**
4. Solve $x_B = B^{-1}b$ for **basic variables** x_B; B is called a **basis**

Example of basic solution

$$P = \{x \mid Ax = b, x \geq \mathbf{0}\}, \quad A = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 \\ 2 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$$

- Choose m linearly independent columns of A, and re-arrange A:

$$B = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 2 & 1 \end{pmatrix}$$

- Set $\bar{x}_4 = \bar{x}_5 = 0$ (i.e., $x_N = \mathbf{0}$).
- Solve $x_B = B^{-1}b = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} -15 \\ -3 \\ 7 \end{pmatrix}$
- Basic solution $\bar{\boldsymbol{x}} = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$. **Basic solution need not be feasible**.

Basic feasible solution (BFS)

Standard form polyhedron $P = \{x \mid Ax = b, x \geq \mathbf{0}\}$, $A \in \mathbb{R}^{m \times n}$, $\operatorname{rank}(A) = m$

A point $\bar{x}$ is a **basic feasible solution** (BFS) if it is a basic solution that is feasible. That is, $\bar{x}$ is a BFS if

- $\bar{x} \geq \mathbf{0}$,
- $\boldsymbol{A}\bar{\boldsymbol{x}} = \boldsymbol{b}$, and
- the columns of $\boldsymbol{A}$ corresponding to non-zero components of $\bar{\boldsymbol{x}}$ are linearly independent

$$\bar{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{pmatrix}, \quad A = \begin{pmatrix} B & N \end{pmatrix}, \quad A\bar{\boldsymbol{x}} = B\boldsymbol{x}_B + N\boldsymbol{x}_N = b \implies \bar{\boldsymbol{x}} = \begin{pmatrix} B^{-1}b \\ \mathbf{0}^{n-m} \end{pmatrix}$$

Feasibility $\implies$ $\boldsymbol{x}_B = B^{-1}b \geq \mathbf{0}$.

Degenerate BFS

- Consider BFS $\bar{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{pmatrix}$ with basis B s.t. $A = (B, N)$
- By definition, $x_N = \mathbf{0}$ and $x_B = B^{-1}b \geq \mathbf{0}$
- BFS $\bar{\boldsymbol{x}}$ is called **degenerate** if some entries of x_B are zero
- Same degenerate BFS can be resulted from different basis

Example of BFS

$$P = \{x \mid Ax = b, x \geq \mathbf{0}\}, \quad A = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 \\ 2 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$$

Basic solution, but not feasible

$$B = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 2 & 1 \end{pmatrix} \implies \begin{pmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{pmatrix} = \begin{pmatrix} -15 \\ -3 \\ 7 \\ 0 \\ 0 \end{pmatrix}$$

Basic feasible solution (BFS)

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} -1 & 0 \\ 0 & -2 \\ 0 & 1 \end{pmatrix} \implies \begin{pmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Degenerate BFS example

$$P = \{x \mid Ax = b, x \geq \mathbf{0}\}, \quad A = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 \\ 2 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$$

Degenerate BFS, with two different partitions (B, N) and (B', N'):

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 1 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & -1 \\ -1 & 0 \\ 0 & 0 \end{pmatrix} \implies \begin{pmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$B' = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -2 & 0 \\ 2 & 1 & 0 \end{pmatrix}, \quad N' = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} \implies \begin{pmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

BFS $=$ extreme point

Theorem

Assume $\text{rank}(\boldsymbol{A}) = m$. A point $\bar{\boldsymbol{x}}$ is an extreme point of polyhedron $\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}$ if and only if it is a basic feasible solution.

Proof: We show it on blackboard, or consult Theorem 8.7 in text.

Thus,

- "extreme point $=$ basic feasible solution (BFS)".
- Remember, from picture optimal solution at extreme points $\Longrightarrow$ optimal solutions at BFS
- Assertion can be proven formally (we are going to do this next)

Representation thm, standard form polyhedron BFS

- $P = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}$ (i.e., polyhedron in standard form)
- $V = \{\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k\}$ be the extreme points of P
- $C = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0},\ \boldsymbol{x} \geq \boldsymbol{0}\}$
- $D = \{\boldsymbol{d}^1, \ldots, \boldsymbol{d}^r\}$ be the extreme directions of C

Representation Theorem (standard form polyhedron)
For $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x} \in P$ iff it is the sum of a convex combination of points in V and a non-negative linear combination of points in D, i.e.

$$\boldsymbol{x} = \sum_{i=1}^{k} \alpha_i \boldsymbol{v}^i + \sum_{j=1}^{r} \beta_j \boldsymbol{d}^j,$$

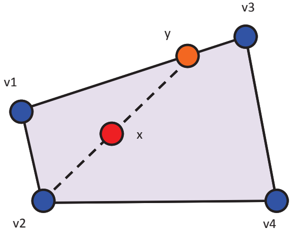
where $\alpha_1, \ldots, \alpha_k \geq 0$, $\sum_{i=1}^{k} \alpha_i = 1$ and $\beta_1, \ldots, \beta_r \geq 0$

Proof: See text Theorem 8.9 (In the proof, Th. 8.9 should be Th. 3.26).

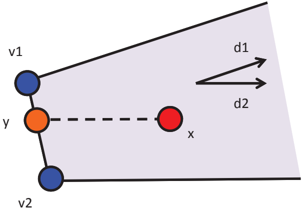
Illustration of representation theorem

Representation theorem provides "inner representation" of polyhedron.

- (a) x is convex combo. of v^2 and y, and y is convex combo. of v^1 and $v^3 \implies x$ is convex combo. of v^1, v^2 and v^3.
- (b) x is convex combo. of v^1 and v^2, plus $\beta_2 d^2$.



(a) Bounded case



(b) Unbounded case

Optimality of extreme points

Now we can present the theorem regarding optimality of extreme points

Theorem

Consider the standard form LP problem

$$\begin{aligned} z^* = \text{infimum} \quad & z = \boldsymbol{c}^T \boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{x} \in P, \end{aligned}$$

(a) This problem has a finite optimal solution if and only if P is nonempty and z is bounded on P, meaning that $\boldsymbol{c}^T \boldsymbol{d}^j \geq \mathbf{0}$ for all $\boldsymbol{d}^j \in D$ (the set of extreme directions of $\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \mathbf{0},\ \boldsymbol{x} \geq \mathbf{0}\}$)

(b) Moreover, if the problem has a finite optimal solution, then there exists an optimal solution among the extreme points.

Proof: We show it on blackboard, or see Theorem 8.10 in text.

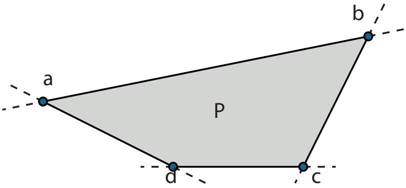
Adjacent BFS's

Two BFS a and b of polyhedron P are **adjacent** if
$\forall\ y \in \alpha a + (1-\alpha)b,\ \alpha \in (0,1)$:

$$y = \lambda u + (1-\lambda)v,\ \ u, v \in P,\ \ \lambda \in (0,1)$$

$$\Longrightarrow \quad \begin{cases} u = \alpha_u a + (1-\alpha_u)b,\ \ \alpha_u \in (0,1) \\ v = \alpha_v a + (1-\alpha_v)b,\ \ \alpha_v \in (0,1) \end{cases}$$

a adjacent to b and d but not c



Algebraic characterization of adjacency

Theorem
Let u and v be two different BFS's corresponding to partitions (B^1, N^1) and (B^2, N^2) respectively. Assume that all but one columns of B^1 and B^2 are the same. Then u and v are adjacent BFS's.

Proof: We show it on blackboard, or see Proposition 8.13 in text.

- Theorem useful in geometric interpretation of simplex algorithm (next lecture).
- A converse of the theorem holds (see text).

Summary

So far, we have seen

- All linear programs can be written in standard form.
- Extreme point = basic feasible solution (BFS).
- If a standard form LP has finite optimal solution, then it has an optimal BFS.

We finally have rationale to search only the BFS's to solve a standard form LP. This is the main characteristic of the **simplex algorithm**.