TMA 965 : Discrete Matematics
D3-spring, 03

Homework1

The following rules are in force for all homework exercises. (see the course description):

1. Explain (prove) all of your answers.
2. Inform me whom you have worked with. (Recall you are encouraged to cooperate with other students to figure out how to solve the problems but the final formulating and writing up of the solutions must be done yourself.)
3. Save (of course) all your homeworks together with my comments.

Warm up problems

1. A group of 8 right-handed and 6 left-handed ping pong players are in a room. In how many ways can they pair off so that only one pair consists of two right-handed players?

2. How many path of minimal length, along the edges, are there from the origin to the point $(1, 1, ..., 1)$ in the unit cube?

3. When the mathematician Gauβ was 9 years old, he was so bored in school that one day he proved the following well-known formula:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

In today's schools, one usually proves this by induction; do this. But Gauβ had another much simpler proof; can you find it?

4. Let S be a finite set. Prove *combinatorially* that the number of subsets of S that have an even number of elements is equal to the number of subsets of S that have an odd number of elements. One can also obtain this as a consequence of the binomial formula but the point is to do this *combinatorially* which means to describe an explicit 1 to 1 correspondence between the two collections.

5 (a). The world's best soccer club, Liverpool FC (!!!), last year had a team consisting of 22 players : 3 goal keepers, 6 defenders, 9 midfield players and 4 offensive players. The trainer prefers the 4-4-2 system which means that there is 1 goal keeper, 4 defenders, 4 midfield players and 2 offensive players. How many different combinations can play a game if (a) the internal order within each category of player matters; (a) the internal order within each category of player does not matter.

To be handed in Wednesday January 28

1. Exercise 3.2.2 in the book, page 49 (1985 edition).
Suppose we have a number of different subsets of $\{1, 2, \ldots, 8\}$, with the property that each one has four members, and each member of $\{1, 2, \ldots, 8\}$ belongs to exactly three of the subsets. How many subsets are there? Write down a collection of subsets which satisfies the conditions.

2. Exercise 3.5.4 in the book, page 55 (1985 edition).
Let $(n)_m = n(n-1)\ldots,(n-m+1)$. By interpreting the result in terms of ordered selections, show that

$$(n)_m(n-m)_{r-m} = (n)_r$$

for any positive integers satisfying $n > r > m$.

3. Exercise 3.7.7 in the book, page 60 (1985 edition).
How many five-digit telephone numbers have a digit which occurs more than once?

4. A family consists of 9, people, 5 men and 4 woman. They are going to play a game where one first must choose 3 referees, 2 who are men and 1 who is a woman. Then the remaining 6 people have to be paired off so that each pair consists of a woman and a man. In how many ways can this be done?

5. Let q_n be the number of words of length n with alphabet $\{a, b, c, d, e, f\}$ that has an odd number of b:s. Prove that

$$q_{n+1} = 6^n + 4q_n.$$

(Hint : Divide up the words of length $n+1$ according to whether they begin with a b or not.).

6. More or less Exercise 4.1.7 in the book, page 66 (1985 edition).
Prove the following identity combinatorially.

$$\binom{s-1}{0} + \binom{s}{1} + \ldots + \binom{s+n-2}{n-1} + \binom{s+n-1}{n} = \binom{s+n}{n}$$

Some exercises from last year

1. An ordered triple (a, b, c) av whole integers is called an *arithmetic progression (AP)* if $b - a = c - b$. Give a formula for the number of AP's that consist of numbers from $\{1, 2, ..., n\}$.

2. Give and prove a simpler expression for

$$\sum_{k=0}^{n} k \cdot \binom{n}{k}$$

OBSERVE! You get more points if your proof does not use the formula for $\binom{n}{k}$; this would mean you give a combinatorial proof.

3. More soccer! In the soccer world championships, there are 32 countries. The 32 teams will be divided into 8 groups, each group having 4 teams. In how many ways, can this be done if the internal order within each group doesn't matter but the order of the groups themselves does matter.

4. Let $n \geq k \geq i$ be three positive whole integers. Prove *combinatorially* that

$$\binom{n}{i}\binom{n-i}{k-i} = \binom{n}{k}\binom{k}{i}. \tag{1}$$

I REPEAT! By a kombinatorially prove, we mean one that avoids formulas for binomial coefficients. Instead, one needs to show that the two sides of the equation count the same number of *things* but in two different ways. The question is what are these *things*.

Further exercises

See homework exercises from previous years. There are links from the homepage.