Makeup Exam for discrete mathematics D3
TMA965
August 19, 2003
Location: V
Time: 8:45-12:45
Jour: Jeff Steif 772 3513
Ingen hjalpmedel : No books or notes
For grades of 3,4,or 5, 12,18 or 24 needed.

1. (5 points).
Consider the ways to give 21 toys to 5 children.
(a). How many ways can this be done if children are distinguishable, toys are indistinguishable and there is a requirement that each child gets at least two toys?
(b). How many ways can this be done if both toys and children are distinguishable and there is a requirement that there are at least 3 children who each get at least one toy?

Solution: (a) We first hand out 2 toys to each children (doesn't matter which toys) leaving us with 11 and then the number of ways to hand out the remaining 11 is (as derived in class) "15 choose 4".

[OBSERVE THAT THE SOLUTION TO 1(b) WAS CHANGED ON OCTOBER 20, 2003 WHEN AN EXTRA FACTOR OF 2 WHICH SHOULD NOT HAVE BEEN THERE WAS REMOVED.]

(b) This would be the total number of ways to hand things out (5^{21}) minus the number of ways to hand things out so that at most 2 people get toys. The number of ways that exactly one person gets any toys is 5. The number of ways that exactly two people get any toys is "5 choose 2" times $2^{21} - 2$ since we have to choose 2 among the 5 people to give the toys to and then we have to choose a subset which is neither empty nor all the toys (there are $2^{21}-2$ such subsets) to give to the first of these two people. Hence we get

$$5^{21} - 5 - C(5,2)[2^{21} - 2].$$

C(5,2) means "5 choose 2".

2. (5 points).
(a). Give an example of a graph which does not contain an Eulerian cycle. (Recall a cycle always ends at the same point that it starts and a Eulerian cycle is a cycle which goes through each edge exactly once.)
(b). What does Euler's theorem say about a sufficient and necessary condition for a graph to have an Eulerian cycle?
(c). Give an example of a graph which has chromatic number 3 and does not contain a triangle. Can you find such an example which is also a bipartite graph?
(d). Give an example of a graph which has exactly two spanning trees.

Solution: (a) A tree which is not simply a path.
(b) There are no vertices of odd degree.
(c) A cycle of length 5. No. All bipartite graphs have chromatic number at most 2.
(d). [FORGET THIS PROBLEM]

3. (5 points).
Consider the set of integers modulo 9000 with modular arithmetic. This is the set $\{0, 1, \ldots, 8999\}$ together with both modular addition and modular multiplication.
(a). How many of these elements have additive inverses?
(b). Describe which elements have multiplicative inverses. How many are there?
(c) How does one compute the multiplicative inverse of an element (assuming it has a multiplicative inverse)?

Solution: (a) All of them.
(b) The elements which have a multiplicative inverse are those which are relatively prime to 9000. The number of these is $\phi(9000)$. Since $9000 = 3^2 2^3 5^3$, $\phi(9000)$ is 9000[2/3][4/5][1/2]=2400.
(c) If a has a multiplicative inverse modulo 9000, then gcd(a,9000)=1. One can then use the Euclidean algorithm (as explained in the book) to find integers x and y so that

$$xa + y9000 = 1.$$

Then x will be the inverse of a since 9000 divides $xa - 1$.

4. (5 points).
(a). What is the cycle representation (i.e., the way of writing the permutation in terms of disjoint cycles) of the following permutation. $\pi(1) = 7, \pi(2) = 5, \pi(3) = 8, \pi(4) = 2, \pi(5) = 4, \pi(6) = 1, \pi(7) = 6$ and $\pi(8) = 3$.
(b). What is the order of the permutation π? In other words, how many times do you need to compose π with itself in order to get the identity permutation?
(c). How many fixed points (1-cycles) does π^2 have? (π^2 is the permutation obtained by composing π with itself.)
(d). Can π be expressed as a product of 3 transpositions? Can π be expressed as a product of 8 transpositions? (Recall that a transposition is a cycle of length 2).

Solution: (a) (176) (254)(38).
(b) The order is 6 which is the least common multiple of the cycle lengths which are 3 and 2.
(c) π^2 has cycle representation (167) (245)(3)(8) and so there are 2 fixed points.
(d) π can be expressed as a product of 5 transpositions (since a cycle of length k can be expressed as a product of $k - 1$ transpositions). Therefore π cannot be expressed as a product of 8 transpositions. It also cannot be expressed as a product of 3 transpositions since a product of 3 transpositions on 8 elements would always have a fixed point since there will always be at least one element which doesn't appear in any of the 3 transpositions.

5. (5 points).
Consider the group G of rigid motions of a square to itself. (The rigid motions include flipping over the square.)
(a). How many elements are there in this group?
(b). Consider the subset H of G consisting of elements which send the left two points to themselves. Is H a subgroup of G? How many elements does H have?
For the rest of the problem, think of G and H as groups of permutations of the 4 corners (or vertices) of the square.
(c). How many elements of G are there in the stabilizer of a fixed corner (or vertex) v? (Recall that the stabilizer of an element is the set of all permutations in the permutation group which fix the element.) How many orbits are there for the corner points when we consider the permutations in G?
(d). How many elements of H are there in the stabilizer of a fixed corner (or vertex) v? How many orbits are there for the 4 corners when we consider the permutations in H?

Solution:

(a). 8 since we first have to decide where the top left corner goes (4 choices) and then once we know where the top left corner goes, there are then two choices.

(b). Yes. 2 elements (the identity and flipping with respect to the horizontal line through the square).

(c) 2. The identity and flipping with respect to the diagonal line through v. There is one orbit since given any vertices v and w, there is a rigid motion taking v to w.

(d) 1. Only the identity. There are two orbits, one being the two left points and one being the two right points. This is because no element of the two left points are moved to one of the two right points by permutations in H but on the other hand we can move the points on the left half to each other and the same for the right half.

6. (5 points).

What is the last digit in the number 7^{111}? (Hint: use modular 10 arithmetic.)

Solution: One first checks that (modulo 10), $7^2 = 9$, $7^3 = 3$ and $7^4 = 1$. Then $7^{111} = 7^{4x27+3} = [7^4]^{27}7^3$. Now, since $7^4 = 1$(modulo 10), we have that (modulo 10) $7^{111} = 1^{27}7^3$, which is 3. Hence the last digit is 3.