Final Exam for discrete mathematics D3

TMA965

October 22, 2003

Location: V

Time: 8:45-12:45

Jour: Jeff Steif 772 3513 or 0702298318

Ingen hjalpmedel : No books, notes or calculators

For grades of 3,4,or 5, 12,18 or 24 needed.

You may use binomial coefficients in your answers. So, for example, expressions like "15 choose 6" meaning $\binom{15}{6}$ do not need to be simplified. If a question has different parts, each part is weighted equally. You may do the exam in English or Swedish. Write clearly and not too small.

1. (5 points).

Consider the ways to give 18 toys to 8 children.

(a). How many ways can this be done if both toys and children are distinguishable and there is no requirement that each child gets at least one toy but there is a requirement that the first child gets exactly 2 toys, the second child gets exactly 3 toys and the third child gets no toys?

(b). How many ways can this be done if children are distinguishable, toys are indistinguishable and there is no requirement that each child gets at least one toy but there is a requirement that the first child gets at least 2 toys?

(c). Let (18, 8) be our Stirling number of the second kind as presented in class. Which question concerning these toys and children would have the answer (18, 8)?

Solution:

1(a). $\binom{18}{2}\binom{16}{3}5^{13}$ since we first have to choose two toys to give to the first child, then, after that, we have to choose three toys from the remaining 16 to give to the second child and then, after that, we can give the remaining 13 toys to any of the last 5 children.

1(b). $\binom{23}{7}$ since we first give the first child 2 toys (doesn't matter which) and then we distribute 16 toys to 8 children in any way we want. The latter number of ways is $\binom{23}{7}$ from class.

1(c). How many ways can we give 18 distinguishable toys to 8 indistinguishable children so that every child gets at least 1 toy?

2. (5 points).

State the max-flow/min-cut theorem. It is not enough to just state the theorem: you need to first define all terms and concepts that appear in the statement of the theorem. You do NOT need to introduce concepts that arise in the *proof* or say anything at all about how the proof of this theorem is done.

Solution:

See book.

3. (5 points).

Consider $\frac{\mathbf{Z}}{2000}$, the set of integers modulo 2000?

(a). How is this object defined and how many elements does it have?

(b). Does [15] have a multiplicative inverse in $\frac{\mathbf{Z}}{2000}$?

(c). Does [7] have a multiplicative inverse in $\frac{\mathbf{Z}}{2000}$?

(d). If you answered yes to either (b) or (c), find the inverses for those you claimed did have an inverse using the Euclidean algorithm.

Solution:

3(a). It is the set of equivalence classes when we take the set of integers and define the equivalence relation where x and y are related if 2000 divides x-y. There are 2000 elements (equivalence classes).

3(b). No, since 15 and 2000 are not relatively prime.

3(c). Yes, since 7 and 2000 are relatively prime.

3(d). The Euclidean algorithm allows one to compute the gcd of 7 and 2000 (which is 1) and at the same time will yield integers x and y such that $7x + 2000y = 1$. (See the book on how you get this x and y). When you apply it, you will get that x =-857 and y=3. Hence $7(-857) + 2000(3) = 1$. Hence the inverse of 7 is -857 or if you prefer, 1143.

4. (5 points).

The following is the exact same set up done in class.

Consider a 4x4 board which has 15 tiles in the 16 places. (The tiles are labelled 1 through 15.] One can move things around only by moving a tile 1 step horizontally or vertically to an adjacent empty square (there is 1 empty square of course). Can you transform the configuration

2	1	7	14
3	10	15	5
9	*	11	12
8	6	13	4

to the configuration

1	7	8	9
3	*	15	11
14	10	12	5
2	4	13	6

?

You need to explain your answer.

* here denotes the empty slot.

Solution:

If we could do this, we would need to make an odd number of moves since we would make one more up move than down move and the same number of right and left moves. On the other hand this permutation is

$$(2,1,7,8)(3)(4,6)(5,11,12)(9,14)(10,*)(13)(15).$$

This is a product of $3+1+2+1+1=8$ transpositions and hence it cannot be written as a product of an odd number of transpositions. Hence it is impossible.

5. (5 points).

(a) Suppose identity cards are manufactured from square cards ruled with a $4x4$ grid, with two of the sixteen squares punched out. How many different cards can be produced in this way? (We consider of course two such punched cards the same if one can be obtained from the other by rotating or flipping over the card).

(b) Suppose identity cards are manufactured from square cards ruled with a $4x4$ grid, with two of the sixteen squares colored red or blue. How many different cards can be produced in this way? (We consider of course two colored cards the same if one can be obtained from the other by rotating or flipping over the card).

Solution:

(a). Let X be the set of cards with 2 holes punched. There are $\binom{16}{2}$ elements in X. There are 8 rigid motions of the square. id, $\sigma, \sigma^2, \sigma^3$ (where σ is rotation by 90 degrees), g_5 and g_6 which are the reflections about the two diagonals, and g_7 and g_8 which are the reflection about the vertical and horizontal lines through the card, If $F(g)$ is the set of fixed elements for g, we get $|F(id)| = \binom{16}{2}, |F(\sigma)| = |F(\sigma^3)| = 0, |F(\sigma^2)| = 8, |F(g_5)| = |F(g_6)| = 12$ and $|F(g_7)| = |F(g_8)| = 8$. By our formula for the number of orbits, we get that the number of orbits is

$$\frac{\binom{16}{2} + 8 + 12 + 12 + 8 + 8}{8} = 21.$$

(b). The set X is now the set of all cards where 2 squares are colored red or blue. X has $4\binom{16}{2}$ elements. The set of transformations are the same as above. It is not hard to verify that with this new X, we have

$|F(id)| = 4\binom{16}{2}, |F(\sigma)| = |F(\sigma^3)| = 0, |F(\sigma^2)| = 16, |F(g_5)| = |F(g_6)| = 36$ and $|F(g_7)| = |F(g_8)| = 16$. By our formula for the number of orbits, we get an answer of 75. The reason that $|F(g_5)| = 36$ is that there are $\binom{4}{2}$ cards with the two holes on the diagonal of reflection and these can be colored in any of 4 ways and there are 6 cards with the two holes not on the diagonal of reflection and these can be colored in any of only 2 ways since the two holes have to be colored the same.

6. (5 points).

(a). Give a combinatorial proof for the equality

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

when $1 < k < n$.

(b). Give a combinatorial proof for the equality

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$

Solution:

We had done this in class. 1(a). Break all k element subsets of $\{1, \ldots, n\}$ into two classes, those not containing 1 and those containing 1. There are $\binom{n-1}{k}$ of the first type and $\binom{n-1}{k-1}$ of the second type.

2(a). There are 3^n sequences of length n with alphabet $1, 2, 3$. Break them up into n classes $C_0, C_1, \ldots, C_n$ where C_k are those sequences where the number of 1's plus the number of 2's is k. $|C_k| = \binom{n}{k} 2^k$ since we first have to choose where the k 1's and 2's together will go (there are $\binom{n}{k}$ choices) and then, after that, we have to decide which of these k locations will be 1's and which will be 2's (there are 2^k choices). Since $3^n = \sum_{k=0}^{n} |C_k|$, the formula is combinatorially proved.