SOLUTIONS FOR Final Exam for discrete mathematics D3

TMA965

March 14, 2003

Location: V

Time: 14:15-18:15

Jour: Jeff Steif 772 3513

Ingen hjalpmedel : No books or notes

1. (10 points).

Consider the ways to give 9 toys to 6 children.

(a). How many ways can this be done if both toys and children are distinguishable and there is no requirement that each child gets at least one toy but there is a requirement that no child gets all 9 toys.

(b). How many ways can this be done if children are distinguishable, toys are indistinguishable and there is no requirement that each child gets at least one toy.

(c). How many ways can this be done if both people and toys are indistinguishable and there IS a requirement that each child gets at least one toy.

(d). Let $S(9,6)$ be our Stirling number of the second kind as presented in class. Which question concerning these toys and children would have the answer $S(9,6)$?

Solution:

[OBSERVE THAT THE SOLUTION TO 1(a) WAS CHANGED ON OCTOBER 20, 2003 WHEN a "1" was changed to a "6".]

(a). $6^9 - 6$.

(b). "14 choose 5".

(c). 3.

(d). In how many ways can 9 distinguishable toys be divided between 6 nondistinguishable children so that every child gets at least one toy?

2. (5 points).

Define (extremely briefly!!! with at most 1 or 2 sentences) the following terms concerning graphs. (You do not need to define what a graph is).

(a). bipartite graph.

(b). chromatic number.

(c). Eulerian path or cycle.

(d). Hamiltonian cycle.

(e). Spanning tree.

(f). What does Euler's theorem say about a sufficient and necessary condition for a graph to have an Eulerian cycle.

Solution:

For (a)-(e), see book. For (f), the number of vertices of odd degree must be 0 or 2.

3. (5 points).

(a). How many elements are there in the set $\frac{\mathbf{Z}}{7000}$, the set of integers modulo 7000?

(b). How many of these have additive inverses?

(c). How many of these have multiplicative inverses?

Solution:

(a) 7000

(b) All.

(c) 2400.

4. (10 points).

(a). What is the cycle representation (i.e., the way of writing the permutation in terms of disjoint cycles) of the following permutation.

$f(1) = 6, f(2) = 7, f(3) = 3, f(4) = 5, f(5) = 4, f(6) = 2, f(7) = 1.$

(b). What is the order of this permutation?

(c). Can this permutation be expressed as a product of 8 transpositions?

Solution:

(a) $(1627)(3)(45)$

(b) 4.

(c) Yes. $(12)(12)(12)(12)(45)(17)(12)(16)$.

5. (15 points).

Consider the group G of rigid motions of a 3-dimensional cube.

(a). How many elements are there in this group?

(b). Consider the subset H of G consisting of elements which send the top half to itself. Is H a subgroup of G?

For the rest of the problem, think of G and H as groups of permutations of the vertices.

(c). How many elements of G are there in the stabilizer of a fixed vertex v? (Recall that the stabilizer of an element is the set of all permutations in the permutation group which fix the element.) How many orbits are there for the vertices of the cube when we consider the permutations in G?

(d). How many elements of H are there in the stabilizer of a fixed vertex v? How many orbits are there for the vertices of the cube when we consider the permutations in H?

Solution:

(a) 24. Since each vertex can be sent to 8 possible places and then we can rotate around that vertex in 3 ways.

(b) Yes.

(c). 3. The rotations around that point. 1. Every vertex can be moved to any other vertex.

(d) 1. Since every nonidentity element in H moves all the vertices. 2. The top elements can be moved to each other and the same with the bottom ones but the top ones cannot be moved to the bottom ones.

6. (15 points).

Let G be a group. Define an equivalence relation R on G by

$$R = \{(a, b) : \text{ there exists } g \in G \text{ such that } gag^{-1} = b\}.$$

(a). Show R is an equivalence relation.

Let H be the union of all the equivalence classes with only 1 element.

(b). Determine H for the two groups $\frac{\mathbf{Z}}{7} \times \frac{\mathbf{Z}}{5}$ and S_3 (S_3 is the group of all permutations of a 3 element set).

(c). Prove that for every group G, H is a subgroup of G.

Solution:

(a) Taking g to be the identity shows that R is reflexive. $(a, b) \in R$ implies there is g such that $gag^{-1} = b$ which implies that $a = g^{-1}bg$ which implies that $(b, a) \in R$ showing symmetry. $(a, b), (b, c) \in R$ implies there is g, h such that $gag^{-1} = b$ and $hbh^{-1} = c$ which implies that $(a, c) \in R$ showing transitivity.

(b) The whole group for the first case and only the identity for the second group.

(c) $a \in H$ if and only if $ag = ga$ for all $g \in G$. Now, $a, b \in H$ implies that $ag = ga$ and $bg = gb$ for all $g \in G$ which implies that $abg = gab$ for all $g \in G$ which gives $ab \in H$. Showing that $a \in H$ implies that $a^{-1} \in H$ is similar.

7. (15 points).

Consider a group with 91 elements. Prove that there exists an element whose order is 7.

Solution:

Case 1: the group is cyclic. Then the group is (isomorphic to) $\mathbf{Z}_{91}$ and then 13 has order 7.

Case 2: the group is not cyclic. Then each nonindentity element has order 7 or 13 by Lagrange's Theorem. If H_1 and H_2 are two subgroups of order 7 or 13, then there intersection (by Lagrange's Theorem) only contains the identity. It follows that the number of elements of order 13 must be $12k$ for some integer k and the number of elements of order 7 must be 6ℓ for some integer ℓ. We then have $91 = 1 + 12k + 6\ell$. Since 12 does not divide 90, ℓ must be positive and we then have an element of order 7.