TMA 965 : Discrete Matematics D3-fall, 03

Homework 2

You are strongly encouraged to do as many of these exercises as you can before the exercise session on September 18. (One learns the material much better by working on the problems rather than just listening to the solutions presented). The second part of the homework may be handed in. The total possible credit for all homeworks handed in is 10 %. It is not mandatory to hand it in.

The following rules are in force for all homework exercises which are to be handed in.

1. Explain (prove) all of your answers.
2. You are strongly encouraged to cooperate with other students to figure out how to solve the problems but the final formulating and writing up of the solutions must be done yourself. Inform me whom you have worked with.
3. Save (of course) all your homeworks together with my comments.
4. Homework after the due date will not be accepted.

Warm up problems

[Although I call them "warm up" problems, they are not necessarily always much easier than the exercises to be handed in. The point is to give you more exercises to practice on so that you can understand the material better.]

1. Let k, n be nonnegative integers. How many sequences of integers $x_1, \ldots, x_n$ are there so that

$$0 \leq x_1 \leq, \ldots, \leq x_n \leq k?$$

2. How many subsets of $\{1, 2, \ldots, n\}$ are there which contain at least one odd element?

3. If we have $2n$ people in a room, in how many ways can they be paired off?

4. A committee is to be chosen from a set of 7 woman and 4 men. How many ways are there to form the committee if
(a) the committee has 5 people, 3 woman and 2 men?
(b) the committee can be any size (other than empty) but it must have an equal number of men and woman?
(c) the committee has 4 people and 1 of them must be Mr. Smith?
(d) the committee has 4 people, 2 of each sex, and Mr. and Ms. Smith cannot both be on the committee?

5 Consider an infinite chess board. How many ways can a particle move so that
(1) it starts at a given location x
(2) it is back at x at time n
(3) each move consists of a horizontal or vertical move of length 1 (so in particular, the particle cannot stay in the same location during a move).

6. Show that the number of different possible positions of tic-tac-toe after 4 moves is 756.

7. Assume that $F_0 = F_1 = 1$ and that $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Show that

$$F_n F_{n+2} = F_{n+1}^2 + (-1)^n$$

for every $n \geq 0$ without explicitly computing the values of F_n.

May be handed in Thursday September 18

1. A store has 3 different types of candies. For one dollar, you can buy a box of 12 candies, combining the 3 different types of candies in any way you wish. How many different types of possible boxes are there?

2. (a) If we have n indistinguishable objects to be given to k distinguishable

people such that at least one person is given no objects, then in how many ways can this be done?
(b) If we have n indistinguishable objects to be given to k distinguishable people such that the first and last person receive at least two objects each, then in how many ways can this be done?

3. The leaders of the US decided that 50 states is too many states and that different states should be combined to form new larger states. They also want that in the end there are at most 5 states.
(a) In how many ways can this be done? Your answer can involve Stirling numbers of the second kind.
(b) Using the basic recurrence relation for these Stirling numbers, give an explicit answer (not involving these Stirling numbers) if we assume that we begin with 8 states instead of 50.

4. We saw in class that the number of ways to distribute n distinguishable objects to k distinguishable people such that everyone gets at least one object is $k!S(n, k)$ where $S(n, k)$ is the Stirling of the second kind (as defined in the book and in class). Here is another way to count the number of ways of doing this. We can think of first giving each of the people one object. This can be done in $n(n-1)\dots(n-k+1)$ ways since the first person has n choices of object, the second person has $n-1$ choices of object, etc. Once everyone has one object, we don't have to worry about the restriction that everyone receives one object and we can freely give the remaining $n-k$ objects to the k people in any way we want. The number of ways of doing this is k^{n-k} since each of the $n-k$ remaining objects can go to any of the k people. Hence the total number of ways of distributing the objects is $[n(n-1)\dots(n-k+1)]k^{n-k}$. Hence we should have that $k!S(n, k) = [n(n-1)\dots(n-k+1)]k^{n-k}$. Is this correct? If so, prove it. If not, explain the paradox.