TMA 965 : Discrete Matematics D3-fall, 03

Homework 3

You are strongly encouraged to do as many of these exercises as you can before the exercise session on September 25. (One learns the material much better by working on the problems rather than just listening to the solutions presented). The second part of the homework may be handed in. The total possible credit for all homeworks handed in is 10 %. It is not mandatory to hand it in.

The following rules are in force for all homework exercises which are handed in.

1. Explain (prove) all of your answers.
2. You are strongly encouraged to cooperate with other students to figure out how to solve the problems but the final formulating and writing up of the solutions must be done yourself. Inform me whom you have worked with.
3. Save (of course) all your homeworks together with my comments.
4. Homework after the due date will not be accepted.

Warm up problems

[Although I call them "warm up" problems, they are not necessarily always much easier than the exercises to be handed in. The point is to give you more exercises to practice on so that you can understand the material better.]

1. Exercise 15.8.1

2. Prove that for any graph G with at least 6 vertices, there is either a cycle of length 3 or $\alpha(G) \geq 3$. (See exercise 4 page 2 for the definition of

$\alpha(G)$.) Try to formulate a general question/conjecture that might generalize this.

3. Exercise 17.1.2

4. Prove that every tree with at least 2 vertices has at least two leaves.

5 Show that for any graph with at least 2 vertices, there are two vertices with the same degree.

6 Let $\delta(G)$ be the minimum degree of a vertex of a graph G. Show that G contains a path of length at least $\delta(G)$. (The length of a path is the number of edges in it.)

7 The distance between two vertices x and y in a graph, denoted $d(x, y)$, is the length of the shortest path between them. The radius of a graph $rad(G)$ is defined to be $\min_{x \in V} \max_{y \in V} d(x, y)$. (Think why it is reasonable to define the radius in this way.) Show that any graph of radius at most k and maximum degree of at most d has no more than $1 + kd^k$ vertices.

May be handed in Thursday September 25

$\chi(G)$ denotes the chromatic number of G.

1. For any graph G,

$$\chi(G) \leq \frac{1}{2} + \sqrt{2|E| + \frac{1}{4}}$$

Hint: Show first that $|E| \geq \frac{1}{2}\chi(G)(\chi(G) - 1)$.

2. A subset V' of the vertices V in a graph $G = (V, E)$ is called *independent* if there is no edge between any two of the vertices in V'. The *independence number* of a graph G, denoted usually $\alpha(G)$, is the size of the largest independent set. Show $\chi(G) \geq \frac{|V|}{\alpha(G)}$.

3-4. In this exercise, you will go through the steps to prove that the complete graph on 5 vertices (this is the graph where all pairs of vertices have an edge between them) is not planar; this means that you cannot draw it on a piece of paper so that the edges do not cross.

Step 1: Prove Euler's formula which says the following. Let G be a connected graph drawn in the plane. Let R denote the number of regions in the complement of the drawn graph including the piece lying outside the graph. (For example, if we draw a square in the plane and then add one more direct edge from the top right corner to the bottom left corner, then $R = 3$). Then $|V| - |E| + R = 2$.

(Observe that for the special case of trees (which can always be drawn in the plane), we recover the fact that the number of vertices is one more than the number of edges.)

Hint: Do this by induction on the number of edges in the graph. (You need not prove certain geometric facts that might arise which seem obvious).

Step 2 If we have a graph drawn in the plane such that each region is surrounded by 3 edges (such a graph is called a triangulation), then $|E| = 3|V| - 6$.

Hint: Consider the set of pairs (e, A) where e is an edge, A is one of the regions in the complement of the graph and e is one of the edges on the boundary of A. Count this set in two ways thereby obtaining a relationship between $|E|$ and R. Then use Euler's formula to obtain $|E| = 3|V| - 6$.

Step 3. Show that for any graph drawn in the plane $|E| \leq 3|V| - 6$.

Hint: One can add edges to the graph so that it becomes a triangulation. (You need not prove certain geometric facts that might arise which seem obvious).

Step 4. Complete the proof that the complete graph on 5 vertices is not planar.

Extra credit Modify the above proof to show that the gas, water and electricity graph is not planar.