Chromatic polynomials

Definition 1 Let G be a finite graph with vertex set V. A *n-coloring of G* is a map $f : V \longrightarrow \{1, 2, \ldots, n\}$ such that $f(x) \neq f(y)$ if x and y are adjacent in G.

Theorem: The chromatic polynomial is a polynomial in n

Definition 2 Let G be a finite graph. The *chromatic polynomial of G* is the function $\chi_G(n)$ = number of n-colorings of G.

Observe that the *chromatic number* of G (the least n for which G has an n-coloring), is equal to the least (positive) integer n for which $\chi_G(n)$ is positive.

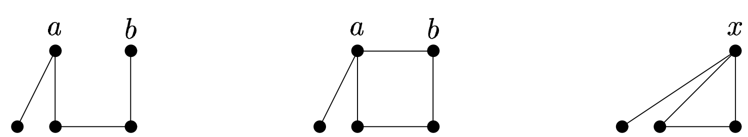
Computing the chromatic polynomial of a graph, or even the chromatic number, is an NP-complete problem.

Here is one way to determine the chromatic polynomial of a graph G:

Theorem 3 *Given a graph G, and two different vertices a, b i G, so that (a, b) is not an edge in G, let G_{a-b} be the graph obtained by adding an edge between a and b in G, and let $G_{a=b}$ be the graph obtained by replacing a and b by a vertex x so that x has an edge to each vertex to which a or b has an edge in G. Then*

$$\chi_G(n) = \chi_{G_{a-b}}(n) + \chi_{G_{a=b}}(n).$$

Proof: Each coloring of G where a and b have different colors corresponds to a unique coloring of G_{a-b} and each coloring of G where a and b have the same color corresponds to a unique coloring of $G_{a=b}$. □



$$\chi_G(n) \quad = \quad \chi_{G_{a-b}}(n) \quad + \quad \chi_{G_{a=b}}(n)$$

Observe that the recursive formula in Theorem 3 can also be written thus:

$$\chi_{G_{a-b}}(n) = \chi_G(n) - \chi_{G_{a=b}}(n). \tag{1}$$

Using this version of the formula means removing edges from the graph until there are only isolated vertices left. Which of these two formulas is more effective in computing χ_G depends on G's structure, in particular on whether it has few or many edges.

Definition 4 Let G be a graph and M a set of vertices in G. The set M is *stable* (or *independent*) if no two vertices in M are adjacent in G.

Definition 5 The i-th *falling factorial of n* is $(n)_i = n(n-1)(n-2)\cdots(n-i+1)$. We let $(n)_0 = 1$.

A coloring of G with exactly k colors always induces a partition of the vertices in G into k stable sets, because each set of like-colored vertices must be stable. Conversely, every partition of G into k stable sets gives a coloring of G with k colors, by assigning one color to each stable set. If we have n colors to choose from then this can be done in $n(n-1)(n-2)\cdots(n-k+1)$ different ways, that is, in $(n)_k$ different ways. This leads to the following theorem.

Theorem 6 *Let $S_G(k)$ be the number of ways to partition the vertices of G into exactly k stable sets. Then*

$$\chi_G(n) = \sum_k S_G(n)_k.$$

It follows directly that $\chi_G(n)$ is a polynomial in n, since $(n)_k$ is obviously a polynomial in n for each k. This also shows that χ_G is monic and of degree $d =$ number of vertices in G.