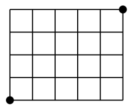
Give arguments to all solutions. List all collaborators.

Warmup



1. In how many ways can you go from $(0, 0)$ to (n, k) in a plane lattice, if you can only go along edges, and each step has to be either right or up?
2. In how many different ways can you place 8 rooks on a chessboard so that none can take another?
3. In how many different ways can we seat 12 people at a round table, if we only care about the relative ordering (who is next to whom)?
4. Show that $\sum_k \binom{n}{k} 2^k = 3^n$, both combinatorially and using the binomial theorem.
5. Assume we should go from the origin to the point $(1, 1, \dots, 1)$ along edges in the d-dimensional unit cube. In how many ways can this be done?
6. Suppose we need to pair off $2n$ people. In how many different ways can this be done?
7. Suppose that n people are to be placed in places numbered from 0 to k. In how many ways can this be done, if every place is allowed to contain any number of people?

To be handed in Tuesday September 19, 13.15 at the latest.

1. (a) Show combinatorially and algebraically that $\binom{n}{k}\binom{k}{i} = \binom{n}{i}\binom{n-i}{k-i}$.

   (b) Simplify $\sum_i \sum_k \binom{n}{i}\binom{n-i}{k-i}$.
2. For a set with $n > 0$ elements, half of its subsets contain an even number of elements and half an odd number of elements. Show this combinatorially, i.e., give an explicit bijection between the two collections. Why does one need that $n > 0$?
3. How many integers $1 \leq k \leq 2006$ are not divisible by neither 2, 3 nor 7? How many are divisible by 7, but neither 2 nor 3?
4. Show that the number of lattice paths with steps of type $(1, 0)$ and $(0, 1)$, which start in $(0, 0)$, end in (n, n) and never rises above the diagonal, is the n-th *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$.
   Hint. Consider paths from $(1, 0)$ to $(n + 1, n)$ which never meet the diagonal and count first the number of paths meeting or crossing the diagonal by reflecting in the diagonal the first part of every path (until it meets the diagonal).

5. Show that the number of ways to multiply $n+1$ symbols is also the n-th *Catalan number* C_n. The order of the multiplications is given by parentheses, in fact it suffices to use only right parentheses, or some other multiplication mark (reverse polish notation).

Bonus problems (no collaboration)

6. Show that the number of ways to divide a convex $(n+2)$-gon into n triangles by drawing $n-1$ non-intersecting diagonals is also the n-th *Catalan number* C_n.

7. Place n points on a circle and draw the chords between each pair of points. Assume that the points are in *general position*, that is, no three chords intersect in a point. Determine the number of regions inside the circle (give a simple formula and prove it combinatorially).
   Hint. Check the first few cases and formulate a guess. Chances are it was wrong. Check another couple of cases and try to write the answer as a simple sum of binomial coefficients.

Further problem

1. How many five digit telephone numbers are there, having at least one digit appearing at least twice?

2. Let n and k be natural numbers. How many sequences of natural numbers $x_1, x_2, \ldots, x_n$ are there such that $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq k$?

3. Simplify $\sum_k k\binom{n}{k}$, preferably using a combinatorial argument.

4. In how many ways can $n+1$ be written as a sum of ones and twos, (where the order matters)? As an example 4 can be written in 5 different ways:

$$1+1+1+1, \quad 1+1+2, \quad 1+2+1, \quad 2+1+1, \quad 2+2.$$

   Having guessed the answer, you may give an inductive proof, or present a bijection with a ”known” set with equally many elements for each n.

5. Give a combiantorial proof of the identity $\sum_{n=k}^{m} \binom{n}{k} = \binom{m+1}{k+1}$.

6. How many different necklaces can you make with 20 pearls, each of a different color? Assume the necklace is “closed,” that is, that it cannot be said to begin at any particular place. (The answer is not equal to $n!$ for any n.)

7. How many of the subsets of the set $\{1, 2, \ldots, n\}$ contain at least one odd number?