Give arguments to all solutions. List all collaborators.

Warmup

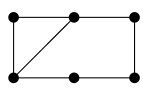
1. Find the chromatic polynomial of a tree with n vertices.
2. Find the chromatic polynomial for the complete graph K_n.
3. Biggs 13.1.1
4. Biggs 13.1.4, 13.1.2
5. What is the last digit in 7^{93}?
6. Biggs 13.6.7
7. (a) Use the binomial theorem (but *not* Fermat's Little Theorem) to show that $(a+b)^p \equiv a^p + b^p \pmod{p}$ if p is prime.

   (b) Show that every prime $p \geq 3$ divides $2^{p-1} - 1$.
8. What is the remainder when we divide 2^{371} by 31?
9. Given a positive integer n, remove the last digit and subtract it twice from what is left. Continue until the result is less than 10. Example: $1603 \to 160 - 2 \cdot 3 = 154 \to 15 - 2 \cdot 4 = 7$. Show that the final result is divisible by 7 if and only if n is.
10. Solve the simultaneous equations $x + 4y = -1$, $3x + 2y = 2$ in $\mathbb{Z}_7$, $\mathbb{Z}_5$ and $\mathbb{Z}_{10}$. In the last case we can subtract two times the second equation from the first, giving $-5x = -5$, or $x = 1$. The first equation then gives that $y = 7 \in \mathbb{Z}_{10}$. Is this solution correct? If not, explain what went wrong with the computation.
11. Show that $\varphi(p^k) = p^k - p^{k-1}$ for a prime p, $k \geq 1$.
12. Biggs 13.6.9

The exercises below are to be handed in
Tuesday October 03, 13.15 at the latest.

1. (a) Determine the chromatic polynomial of the n cycle. Hint: Induction.

   (b) Find the chromatic polynomial of the graph in the margin. Check that your answer gives the right values (but don't hand in those computations): $0, 0, 0, 30,\ 480, 3060, 12480, \ldots$

   You may also think about why all these values are divisible by 30.

2. (a) Show that if $x = x^{-1}$ in $\mathbb{Z}_p^*$, where p is prime, then $x^2 - 1 = 0$ in $\mathbb{Z}_p$. Also show that this proves that 1 and $p-1$ are the only elements in $\mathbb{Z}_p^*$ that are their own inverses.

   (b) Let p be a prime. Show, by looking at the product of all elements in $\mathbb{Z}_p^*$, that $(p-1)! \equiv -1 \pmod{p}$.

3. Let p och q be primes greater than 3. Show that $p^2 - q^2$ is divisible by 24.

4. The numbers $F_n = 2^{2^n} + 1$, $n = 0, 1, 2, \ldots$, are called Fermat numbers. $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$ and $F_4 = 65537$ are all prime. Fermat claimed that all F_n are prime, but Euler showed that 641 divides F_5. Show this by computing in $\mathbb{Z}_{641}$, using $5 \cdot 2^7 + 1 = 641$ and $5^4 + 2^4 = 641$.

5. The numbers $1, 2, \ldots, 20$ are written on the blackboard. Two players take turns erasing one of the numbers, until there are two left. If the sum of these two numbers is divisible by 3, the person who made the last move wins. Who wins, and how?

Bonus problem (no collaboration)

6. Let G be the complement to the graph $\bullet - \bullet - \bullet - \cdots - \bullet - \bullet - \bullet$. Find G's chromatic polynomial.