TMA970

Inledande matematisk analys F/TM

Lösningar 12/11-2012

1. (a) divergent; (b) konvergent;
(c) konvergent; (d) divergent;
(e) falskt; (f) falskt;
(g) falskt; (h) sant.

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2. (a) $\left(\sqrt{1-x^2}\right)^{1/x^4} = e^{\frac{1}{x^4}\ln\sqrt{1-x^2}} =$

$$= e^{\frac{1}{2}\underbrace{\frac{\ln(1-x^2)}{-x^2}}_{\to 1}\cdot\underbrace{\frac{1}{x^2}}_{\to\infty}\cdot(-1)} \longrightarrow \text{"}e^{-\infty}\text{"} = 0$$

$(1-x^2>0$ för x nära $0)$

(b) $$\frac{\ln(\cos x)}{e^{2x^2}-\cos x} =$$

$$= \frac{\ln(1+(\cos x-1))}{\cos x-1}\cdot\frac{\cos x-1}{(e^{2x^2}-1)-(\cos x-1)} =$$

$$= \frac{1}{\frac{e^{2x^2}-1}{2x^2}\cdot\frac{2x^2}{\cos x-1}-1}\cdot\frac{\ln(1+(\cos x-1))}{\cos x-1} =$$

$$= \frac{1}{\frac{e^{2x^2}-1}{2x^2}\cdot\frac{2x^2}{\underbrace{\cos^2 x-1}_{=-\sin^2 x}}\cdot(\cos x+1)-1}\cdot\frac{\ln(1+(\cos x-1))}{\cos x-1}$$

$$\longrightarrow \frac{1}{1\cdot(-2)\cdot 2-1}\cdot 1 = -\frac{1}{5}$$

(3.) $D_f: x \neq 1$ (2)

$f = 0$ omm $x = -1$

$f > 0 \quad \forall x \in (1, \infty)$

$f < 0 \quad \forall x \in (-\infty, -1) \cup (-1, 1)$

ej jämn / udda, ej periodisk, inga symmetrier

$$\lim_{x \to \pm\infty} f(x) = \text{"}\frac{+\infty}{\pm\infty}\text{"} \quad ???$$

$$\frac{(x+1)^2}{x-1} = \frac{x^{\cancel{2}}\left(1+\frac{1}{x}\right)^2}{\cancel{x}\left(1-\frac{1}{x}\right)} \xrightarrow[x \to \pm\infty]{} \pm\infty$$

$$\left.\begin{aligned} \lim_{x \to +1_-} f(x) &= \text{"}\frac{4}{-0}\text{"} = -\infty \\ \lim_{x \to 1_+} f(x) &= \text{"}\frac{4}{+0}\text{"} = +\infty \end{aligned}\right\} \Rightarrow x = 1 \text{ vertikal asymptot}$$

I $\pm\infty$? $\quad \dfrac{f(x)}{x} = \dfrac{(x+1)^2}{x^2 - x} \xrightarrow[x \to \pm\infty]{} 1$

$$f(x) - x = \frac{x^2+2x+1}{x-1} - x = \frac{\cancel{x^2}+2x+1-\cancel{x^2}+x}{x-1} =$$

$$= \frac{3x+1}{x-1} \xrightarrow[x \to \pm\infty]{} 3$$

$\Rightarrow y = x + 3$ sned asymptot i $\pm\infty$

$$f'(x) = \frac{2(x+1)\cdot(x-1) - (x+1)^2}{(x-1)^2} = \cancel{\ }$$

$$= \frac{2x^2 - 2 - x^2 - 2x - 1}{(x-1)^2} = \frac{x^2 - 2x - 3}{(x-1)^2} =$$

$$= \frac{(x-1)^2 - 4}{(x-1)^2} = \frac{(x-3)(x+1)}{(x-1)^2}$$

(bättre att bryta ut $(x+1)$)

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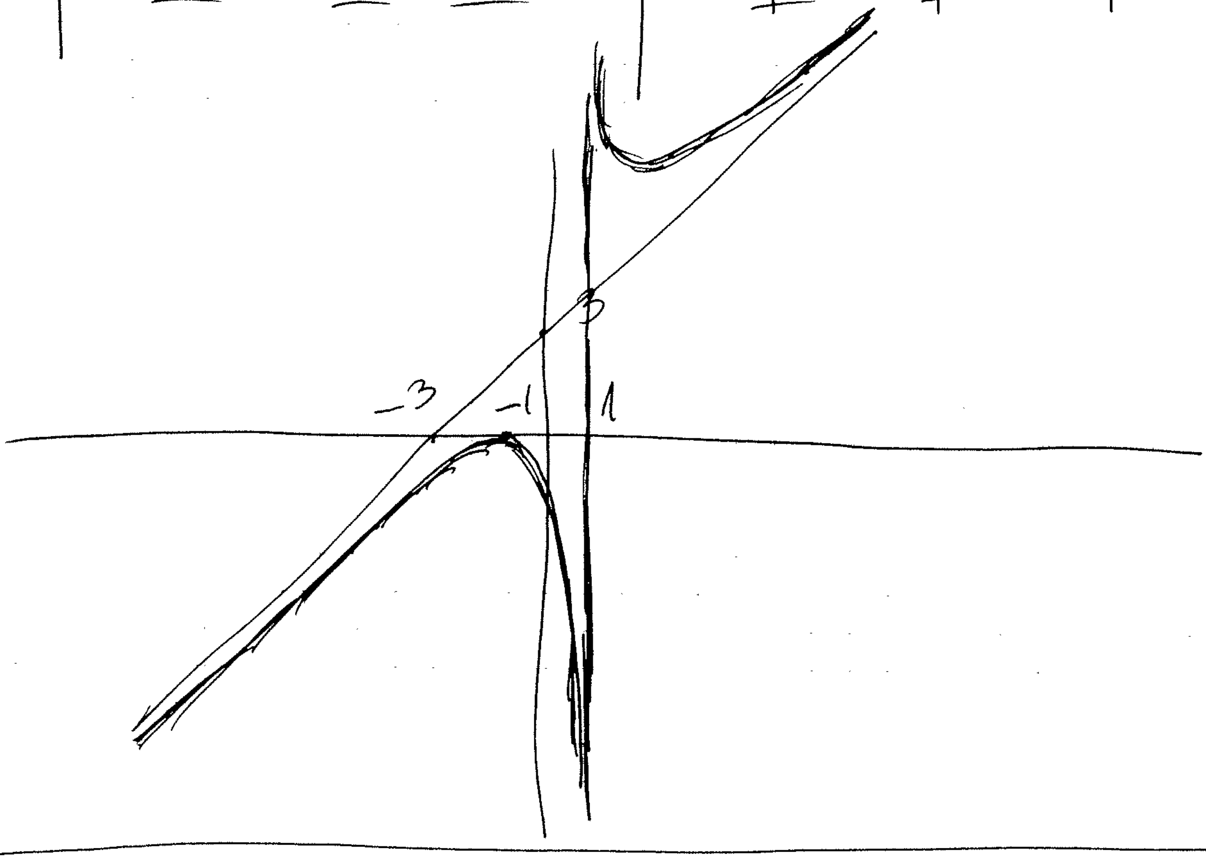
x		-1		1		3	
f'	$+$	0	$-$	|	$-$	0	$+$
		f: lok.max				f: lok.min	

$$f''(x) = \frac{8}{(x-1)^3} \neq 0 \Rightarrow \text{inga inflexionspkter}$$

x	$-\infty$		-1		1		3		$+\infty$
f	$y=x+3$, $-\infty$	$\nearrow$	0 lok. max	$\searrow$ $-\infty$	| $+\infty$	$\searrow$	8 lok. min	$\nearrow$	$y=x+3$, $+\infty$
f'	$+$	$+$	0	$-$ $-$	| $-$ $-$		0	$+$	$+$
f''	$-$	$-$	$-$		|	$+$	$+$		$+$



4. (a) $\int \arctan\sqrt{\frac{1+x}{1-x}}\,dx = x\arctan\sqrt{\frac{1+x}{1-x}} -$

$$- \int x \cdot \frac{1}{1+\frac{1+x}{1-x}} \cdot \frac{1}{2\sqrt{\frac{1+x}{1-x}}} \cdot \frac{\cancel{1}\cancel{-x}+1\cancel{+x}}{(1-x)^2}\,dx =$$

$$= x\arctan\sqrt{\frac{1+x}{1-x}} - \int x \cdot \frac{\cancel{1-x}}{2} \cdot \sqrt{\frac{1-x}{1+x}} \cdot \frac{1}{(1-x)^{\cancel{2}}}\,dx =$$

$$= x\arctan\sqrt{\frac{1+x}{1-x}} - \frac{1}{2}\cdot\frac{1}{2}\int\frac{2x}{\sqrt{1-x^2}}\,dx =$$

$$= x\arctan\sqrt{\frac{1+x}{1-x}} + \frac{1}{4}\cdot 2\int\frac{(1-x^2)'}{2\sqrt{1-x^2}}\,dx =$$

$$= x\arctan\sqrt{\frac{1+x}{1-x}} + \frac{1}{2}\sqrt{1-x^2} \quad (+C)$$

(b) $$\int_1^\infty \frac{\sin x^2}{x^\alpha}\,dx = \lim_{p\to\infty}\int_1^p \frac{\sin x^2}{x^\alpha}\,dx$$

$$\int_1^p \frac{\sin x^2}{x^\alpha}\,dx = \frac{1}{2}\int_1^p \frac{2x\sin x^2}{x^{\tilde\alpha}}\,dx = \left[\begin{array}{l} t = x^2 \\ dt = 2x\,dx \end{array}\right] \qquad \boxed{\tilde\alpha = \alpha+1}$$

$$= \frac{1}{2}\int_1^{p^2}\frac{\sin t}{t^{\tilde\alpha/2}}\,dt \qquad (p^2 \longrightarrow \infty)$$

$$\frac{\tilde\alpha}{2} > 1 \iff \tilde\alpha > 2 : \quad \left|\frac{\sin t}{t^{\tilde\alpha/2}}\right| \le \frac{1}{t^{\tilde\alpha/2}}$$

$\Rightarrow$ konvergent enl. jämförelsekriteriet

$$\int_1^{p^2}\frac{\sin t}{t^{\tilde\alpha/2}}\,dt = \underbrace{\left[\frac{-\cos t}{t^{\tilde\alpha/2}}\right]_1^{p^2}}_{=\cos 1 - \frac{\cos p^2}{p^{\tilde\alpha}} \xrightarrow[p\to\infty]{} 0 \text{ för } \tilde\alpha > 0} + \underbrace{\int_1^{p^2}\frac{\cos t\,\left(-\frac{\tilde\alpha}{2}\right)}{t^{\frac{\tilde\alpha}{2}+1}}\,dt}_{\text{konvergent för } \tilde\alpha > 0 \text{ enl. jämförelsekr.}}$$

$\boxed{\begin{array}{c}\tilde\alpha > 0 \\ \iff \alpha > -1\end{array}}$ konv.

$\boxed{\begin{array}{c}\tilde\alpha \le 0 \\ \iff \alpha \le -1\end{array}}$: divergent (potensen hamnar i täljaren, eller $\int \sin/\cos$)

($\alpha = 0$: konvergent ; ej elementärt att visa det direkt (Fresnelintegral) $\int_0^\infty \left\{\begin{array}{c}\sin\\ \cos\end{array}\right\} x^2\,dx$

5. $x=1$: $0=0=0$

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$f(x) = x-1-\ln x$

$f'(x) = 1-\frac{1}{x} \geq 0 \quad \forall x \geq 1$

$f(1) = 0$ f växande i $[1,\infty)$

$\Rightarrow f(x) \geq 0 \quad \forall x \in [1,\infty)$

$\Rightarrow \ln x \leq x-1$ i $[1,\infty)$

$g(x) = \ln x - 1 + \frac{1}{x}$

$g'(x) = \frac{1}{x} - \frac{1}{x^2} \geq 0$ i $[1,\infty)$

$g(1) = 0$ g växande i $[1,\infty)$

$\Rightarrow \ln x \geq 1-\frac{1}{x}$ i $[1,\infty)$

$g'(x) = \frac{1}{x} - \frac{1}{x^2} \leq 0$ i $(0,1)$

$\Rightarrow g$ avtagande i $(0,1)$

$g(1) = 0 \Rightarrow g \geq 0$ i $(0,1)$

$f'(x) = 1-\frac{1}{x} \leq 0$ i $(0,1)$

$\Rightarrow f$ avtagande i $(0,1]$

$f(1) = 0 \Rightarrow f \geq 0$ i $(0,1]$

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6. $f(x) = \frac{1}{(x-a)^4} + \frac{2}{(x-b)^9}$

$\lim_{x\to a_+} f(x) =$ " $\frac{1}{+0}$ + ändligt tal " $= +\infty$

$$\lim_{x \to b_-} f(x) = \text{"ändligt tal} + \frac{2}{-0}\text{"} = -\infty$$ 6

f kontinuerlig

$f(a+\varepsilon) > 0$, $f(b-\varepsilon) < 0$

för tillräckligt litet $\varepsilon > 0$

$\Rightarrow$ f har nollställe i

$(a+\varepsilon, b-\varepsilon) \subset (a, b)$,

enligt satsen om mellanliggande värde