TMA970 Inledande matematisk analys F/TM

Lösningar 24/10 - 12

1. (a) konvergent; (b) konvergent; (c) konvergent; (d) divergent; (e) sant; (f) falskt; (g) falskt; (h) sant.

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2. (a)
$$\lim_{x\to-\infty}\left(\sqrt{x^2+3x}+x\right)=\lim_{x\to-\infty}\frac{\cancel{x^2}+3x-\cancel{x^2}}{\sqrt{x^2+3x}-x}$$

$$=\lim_{x\to-\infty}\frac{3x}{\underbrace{|x|}_{=-x \text{ ty } x<0}\sqrt{1+\frac{3}{x}}-x}=-\lim_{x\to-\infty}\frac{3\cancel{x}}{\cancel{x}\left(\sqrt{1+\frac{3}{x}}+1\right)}=-\frac{3}{2}$$

(b)
$$\frac{\sqrt{1+x^2}-\cos x}{x\sin 2x}=\frac{\left(\sqrt{1+x^2}\right)^2-\cos^2 x}{x\sin 2x\left(\sqrt{1+x^2}+\cos x\right)}$$

$$=\frac{1+x^2-\cos^2 x}{x\sin 2x\left(\sqrt{1+x^2}+\cos x\right)}=\frac{\sin^2 x+x^2}{2x\sin x\cos x\left(\sqrt{1+x^2}+\cos x\right)}$$

$$=\frac{\sin^{\cancel{2}} x}{2x\cancel{\sin x}\cos x\left(\sqrt{1+x^2}+\cos x\right)}+\frac{x^{\cancel{2}}}{2\cancel{x}\sin x\cos x\left(\sqrt{1+x^2}+\cos x\right)}$$

$\frac{\sin x}{x}\to 1$ $(x\to 0)$, $\cos x\to 1$, $\sqrt{1+x^2}+\cos x\to 2$; $\frac{x}{\sin x}\to 1$, $\cos x\to 1$, $\sqrt{1+x^2}+\cos x\to 2$

$$\xrightarrow[x\to 0]{}\ \frac{1}{2\cdot 2}+\frac{1}{2\cdot 2}=\frac{1}{2}$$

(3.) D_f : alla intervall, i vilka $\cos x > 0$ 2

$\Rightarrow \quad \ldots \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \ldots$

f periodisk, ty $f(x+2\pi) =$

$= \ln \frac{1}{\cos(x+2\pi)} = \ln \frac{1}{\cos x} = f(x)$

(period 2π)

- f jämn, ty $f(-x) = \ln \frac{1}{\cos(-x)} =$

- $= \ln \frac{1}{\cos x} = f(x)$

$\Rightarrow$ det räcker att rita grafen i intervallet $\left[0, \frac{\pi}{2}\right)$; jämn $\Rightarrow$ man kan per symmetri rita den i $\left(-\frac{\pi}{2}, 0\right)$; periodisk $\Rightarrow$ man kan per periodicitet rita den i övriga intervall.

- $\frac{1}{\cos x} \geq 1 \Rightarrow f(x) \geq 0$

  ($f_{min} = 0 = f(0)$)

- globalt minimum

$f(x) = \ln \frac{1}{\cos x} = -\ln \cos x$

$$\lim_{x \to \frac{\pi}{2}-} f(x) = -\lim_{x \to \frac{\pi}{2}-} \left(\underbrace{\ln \underbrace{\cos x}_{\to 0}}_{\to -\infty}\right) = +\infty$$

$\Rightarrow$ vertikal asymptot $x = \frac{\pi}{2}$

(samt $x = (2k+1)\frac{\pi}{2}$)
$k \in \mathbb{Z}$

$$f'(x) = -\frac{1}{\cos x}\cdot(-\sin x) = \tan x$$ 3

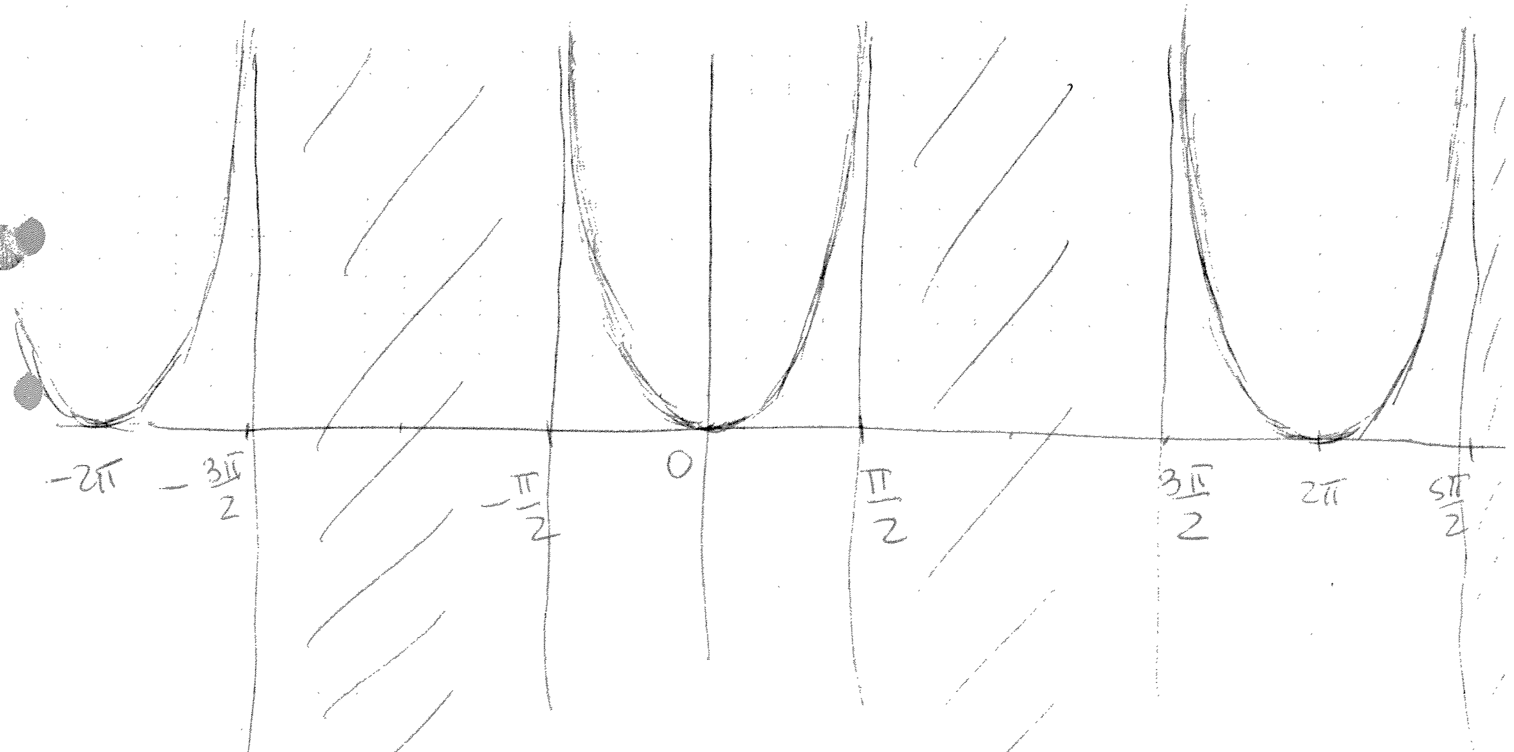
$\Rightarrow f' > 0$ i $\left(0, \frac{\pi}{2}\right)$

$f' < 0$ i $\left(-\frac{\pi}{2}, 0\right)$; $f'(0) = 0$

$\Rightarrow f$ har lokalt min i 0
(vilket vi visste redan)

$$f''(x) = \frac{1}{\cos^2 x} > 0 \Rightarrow f \text{ (strängt) konvex}$$
i varje intervall

	...		$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$		$\frac{3\pi}{2}$	2π ...
f	upprepas		$+\infty$ $\searrow$	0	$\nearrow$ $+\infty$		upprepas	
f'			$-$	0	$+$			
f''			$+$	$+$	$+$			



4. (a) $\displaystyle \int x^2(\ln x)^2\,dx = \frac{1}{3}\int (x^3)'(\ln x)^2\,dx =$ $\boxed{4}$

$\displaystyle = \frac{1}{3}x^3(\ln x)^2 - \frac{1}{3}\int x^{\cancel{3}2}\, 2\ln x \cdot \frac{1}{\cancel{x}}\,dx =$

$\displaystyle = \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\cdot\frac{1}{3}\int (x^3)'\ln x\,dx =$

$\displaystyle = \frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3\ln x + \frac{2}{9}\int x^{\cancel{3}2}\frac{1}{\cancel{x}}\,dx =$

$\displaystyle = \frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3\ln x + \frac{2}{27}x^3 \ (+C)$

(b) $\displaystyle \int_0^p \frac{dx}{\sqrt{e^x+1}} = \left[\begin{array}{l|l} t = e^x & dx = \frac{1}{t}\,dt \\ dt = e^x\,dx & x=0:\ t=1 \\ & x=p:\ t=e^p \end{array}\right] =$

$\displaystyle = \int_1^{e^p} \frac{dt}{t\sqrt{t+1}} = \left[\begin{array}{l|l} t+1 = u^2 \ (u\geq 0) & t=1: u=\sqrt{2} \\ t = u^2-1;\ dt = 2u\,du & t=e^p: u=\sqrt{e^p+1} \end{array}\right]$

$\displaystyle = \int_{\sqrt{2}}^{\sqrt{e^p+1}} \frac{2\cancel{u}\,du}{(u^2-1)\cdot\cancel{u}} = \circledast$

$\displaystyle \frac{1}{u^2-1} = \frac{1}{(u+1)(u-1)} = \frac{A}{u+1} + \frac{B}{u-1}$

$1 = A(u-1) + B(u+1)$

$u=1: \quad 1 = 2B \Rightarrow B = \frac{1}{2}$

$u=-1: \quad 1 = -2A \Rightarrow A = -\frac{1}{2}$

$\displaystyle \Rightarrow \circledast = \cancel{2}\cdot\frac{1}{\cancel{2}}\Big[\ln|u-1|\Big]_{\sqrt{2}}^{\sqrt{e^p+1}} - \cancel{2}\cdot\frac{1}{\cancel{2}}\Big[\ln|u+1|\Big]_{\sqrt{2}}^{\sqrt{e^p+1}} =$

$$\overset{u>1}{=} \left[\ln \frac{u-1}{u+1} \right]_{\sqrt{2}}^{\sqrt{e^p+1}} =$$

(5)

$$= \ln \underbrace{\left(\frac{\sqrt{e^p+1}-1}{\sqrt{e^p+1}+1} \right)}_{\to 1,\ p\to\infty} - \ln \frac{\sqrt{2}-1}{\sqrt{2}+1} \xrightarrow[p\to\infty]{}$$

$\ln(\ldots) \to 0$, $p \to \infty$

$$\longrightarrow 0 + \ln \frac{\sqrt{2}+1}{\sqrt{2}-1} =$$

$$= \ln \frac{(\sqrt{2}+1)^2}{2-1} = \ln(3+2\sqrt{2})$$

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5. D_f : basen måste vara >0

$$1+\frac{1}{x} > 0 \iff \frac{x+1}{x} > 0$$

gäller i $(-\infty,-1) \cup (0,\infty)$

$\lim_{x\to\pm\infty} f(x) = e \Rightarrow y = e$ horisontell asymptot i $\pm\infty$

$$\lim_{x\to 0_+} \left(1+\frac{1}{x}\right)^x = \text{"}+\infty^{+0}\text{"} \quad ?; = \circledast$$

$$\lim_{x\to -1_-} \left(1+\frac{1}{x}\right)^x = \text{"}+0^{-1}\text{"} = +\infty$$

$$\circledast = \left(1+\frac{1}{x}\right)^x = e^{x\ln\left(1+\frac{1}{x}\right)}$$

$$x\ln\left(1+\frac{1}{x}\right) = x\ln\frac{x+1}{x} = \underbrace{x\ln(x+1)}_{0\cdot 0} - \underbrace{x\ln x}_{0}$$

ty $x\ln x = \left[x = \frac{1}{t}\right] = \frac{-\ln t}{t} \xrightarrow[t\to\infty]{} 0$

$$\Rightarrow \circledast = e^0 = 1$$

6. (a) ? $\exists \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$

$$\frac{f(x)-f(c)}{x-c} = f'(\xi_x), \text{ där } \xi_x$$

ligger mellan x o c

$x \to c \Rightarrow \xi_x \to c$ (de två poliserna)

$$\Rightarrow f'(\xi_x) \to \lim_{x \to c} f'(x)$$

(som finns - givet)

$$\Rightarrow \exists f'(c) = \lim_{x \to c} f'(x)$$

(b) $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$\exists f'(0) = 0$, $\nexists \lim_{x \to 0} f'(x)$

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7. (b) ? $\forall \varepsilon > 0 \ \exists A : \forall x > A \ \left| \frac{x+1}{x} - 1 \right| < \varepsilon$

$$\left| \frac{x+1}{x} - 1 \right| = \left| \frac{\cancel{x}+1-\cancel{x}}{x} \right| = \frac{1}{x} < \varepsilon \Leftrightarrow$$

ty $x \to +\infty$

$$\Leftrightarrow x > \frac{1}{\varepsilon}$$

Tag $A = \frac{1}{\varepsilon}$