TMA970 Inledande matematisk analys F/TM

Lösningar 26/10-2013

1 (a) konvergent ; (b) divergent ;
(c) konvergent ; (d) konvergent
(d) sant ; (e) sant ;
(f) falskt ; (g) falskt

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2. (a) $\sqrt{x}\left(\sqrt{x+1}-\sqrt{x}\right) \xrightarrow[x\to\infty]{} \text{"}\infty\cdot(\infty-\infty)\text{"}?$

$$\sqrt{x}\left(\sqrt{x+1}-\sqrt{x}\right) =$$

$$= \frac{\sqrt{x}\left(\sqrt{x+1}-\sqrt{x}\right)\left(\sqrt{x+1}+\sqrt{x}\right)}{\sqrt{x+1}+\sqrt{x}} =$$

$$= \frac{\sqrt{x}\left(\cancel{x}+1-\cancel{x}\right)}{\sqrt{x+1}+\sqrt{x}} =$$

$$= \frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}}\cdot\frac{1}{\sqrt{1+\underbrace{\frac{1}{x}}_{\to 0}}+1} \xrightarrow[x\to\infty]{} \frac{1}{2}$$

(b) $$\frac{\sqrt{1-\cos x^2}}{1-\cos x} = \frac{\sqrt{2\sin^2\frac{x^2}{2}}}{2\sin^2\frac{x}{2}} = \left(\begin{matrix}\sin\frac{x^2}{2}>0\\ \text{när } x\to 0\end{matrix}\right)$$

$$= \frac{1}{\sqrt{2}}\,\frac{\sin\frac{x^2}{2}}{\sin^2\frac{x}{2}} = \frac{1}{\sqrt{2}}\,\frac{\sin\frac{x^2}{2}}{\frac{x^2}{2}}\cdot\frac{\cancel{x^2}}{2}\cdot\frac{\left(\frac{x}{2}\right)^2}{\sin^2\frac{x}{2}}\cdot\frac{1}{\left(\frac{\cancel{x}}{2}\right)^2}$$

$$= \frac{1}{\sqrt{2}}\cdot\frac{1}{\cancel{2}}\cdot\cancel{4}^{2}\cdot\frac{\sin\frac{x^2}{2}}{\frac{x^2}{2}}\cdot\left(\frac{x/2}{\sin\frac{x}{2}}\right)^2 \xrightarrow[x\to 0]{} \sqrt{2}$$

(3) $f(x) = \dfrac{\sin x}{2+\cos x}$ $\triangle 2$

$D_f = \mathbb{R}$, ty $2+\cos x \geq 1 \quad \forall x \in \mathbb{R}$

f periodisk med period 2π

$$f(-x) = \frac{\sin(-x)}{2+\cos(-x)} = \frac{-\sin x}{2+\cos x} = -f(x)$$

$\Rightarrow f$ udda

Inga asymptoter ($D_f = \mathbb{R}$; periodisk)

$f(x) = 0$ för $x = k\pi$, $k \in \mathbb{Z}$

$$\left.\begin{array}{ll} f(x) \geq 0 & \text{i } [2n\pi, \pi + 2n\pi] \\ \leq 0 & \text{i } [\pi + 2n\pi, 2(n+1)\pi] \end{array}\right\} n \in \mathbb{Z}$$

Det räcker att rita grafen i $[0, \pi]$. M.h.a. att f är udda kan vi sedan få grafen i $[-\pi, 0]$, och m.h.a. periodiciteten i hela $\mathbb{R}$.

$$f'(x) = \frac{\cos x(2+\cos x) + \sin^2 x}{(2+\cos x)^2} = \frac{2\cos x + 1}{(2+\cos x)^2}$$

$f'(x) = 0$ för $\cos x = -\frac{1}{2}$,

d.v.s. $x = \frac{2\pi}{3}$ (OBS! i $[0, \pi]$ enl. ovan)

	0		$2\pi/3$		π
f'	$\frac{1}{3}$	$+$	$+\ 0$	$-$	-1

$$f''(x) = \frac{-2\sin x\,(2+\cos x)^{\cancel{2}} + 2\cancel{(2+\cos x)}(\sin 2x + \sin x)}{(2+\cos x)^{\cancel{4}\,3}}$$

$$= \frac{-\cancel{4}2\sin x - \cancel{\sin 2x} + \cancel{2}\sin 2x + 2\cancel{\sin x}}{(2+\cos x)^3} =$$

$$= \frac{\sin x\,(2\cos x - 2)^{\leq 0\ \forall x}}{(2+\cos x)^2_{>0\ \forall x}}$$

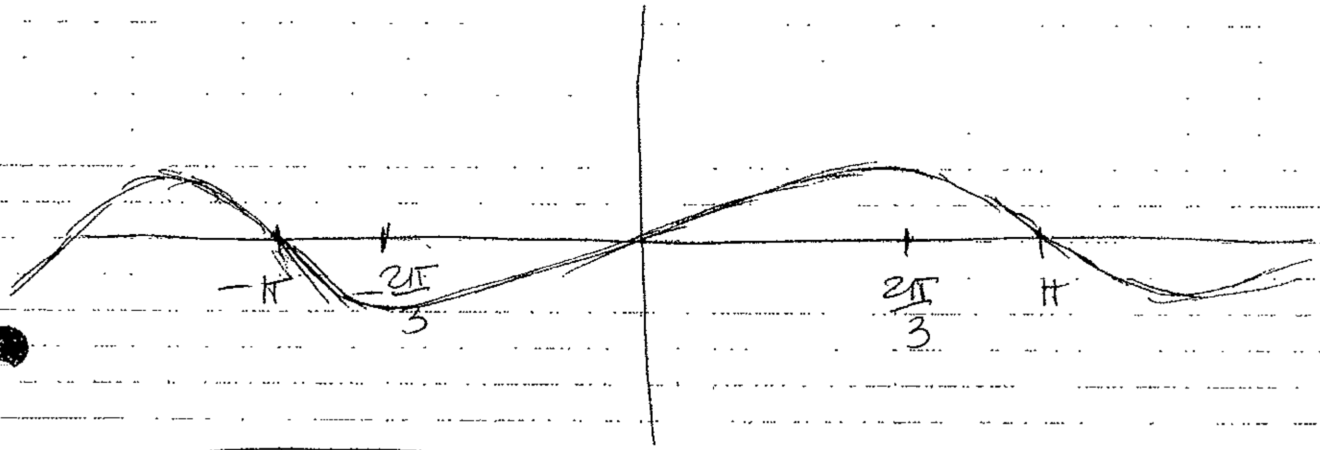
(3)

$f''(x) = 0$ för $x = k\pi$

$f'' < 0$ i $(0, \pi)$

	$-\pi$		$-2\pi/3$		0		$2\pi/3$		π
f	0 infl.	↘	$-\sqrt{3}/3$ lok. min	↗	0 infl.	↗	$\sqrt{3}/3$ lok. max	↘	0 infl.
f'	-1	$-$	0	$+$	$1/3$	$+$	0	$-$	-1
f''	0	$+$		$+$	0	$-$		$-$	0

(tabellen är gjord i $[-\pi, \pi]$ för att visa hur informationen överförs)



(4.) (a) $\int \frac{\cos x}{\sqrt{2+\cos 2x}}\,dx =$

$$= \int \frac{(\sin x)'}{\sqrt{2+1-2\sin^2 x}}\,dx = \left[\begin{array}{l} t = \sin x \\ dt = \cos x\,dx \end{array}\right] =$$

$$= \int \frac{dt}{\sqrt{3-2t^2}} = \frac{1}{\sqrt{3}} \int \frac{dt}{\sqrt{1-\frac{2}{3}t^2}} = \frac{1}{\sqrt{3}} \int \frac{dt}{\sqrt{1-\left(\sqrt{\frac{2}{3}}t\right)^2}} =$$

(4)

$$= \frac{1}{\sqrt{3}} \cdot \sqrt{\frac{3}{2}} \int \frac{\sqrt{\frac{2}{3}}\, dt}{\sqrt{1-\left(\sqrt{\frac{2}{3}}t\right)^2}} =$$

$$= \frac{1}{\sqrt{2}} \arcsin\left(\sqrt{\tfrac{2}{3}}t\right)(+C) =$$

$$= \frac{1}{\sqrt{2}} \arcsin\left(\sqrt{\tfrac{2}{3}}\sin x\right)(+C)$$

(b) $I = \int_{1/e}^{e} |\ln \underbrace{|x|}_{=x}|\, dx = \int_{1/e}^{1} (-\ln x)\, dx + \int_{1}^{e} \ln x\, dx$

$$\int \ln x\, dx = x\ln x - \int x \cdot \frac{1}{x}\, dx =$$
$$= x\ln x - x \quad \text{(räcker med \underline{en} primitiv)}$$

$$\Rightarrow I = -\left[x\ln x\right]_{1/e}^{1} + \left[x\right]_{1/e}^{1} + \left[x\ln x\right]_{1}^{e} - \left[x\right]_{1}^{e} =$$

$$= -0 + \frac{1}{e}\cdot(-1) + 1 - \frac{1}{e} + \cancel{e}\cdot 1 - 0 - \cancel{e} + 1 =$$
$$= 2(1 - 1/e)$$

(5.) $D_f: \left|\frac{2x}{1+x^2}\right| \leq 1$

$$\Leftrightarrow 2|x| \leq 1+x^2 \Leftrightarrow (1-|x|)^2 \geq 0$$

$$\Rightarrow D_f = \mathbb{R};\ f \text{ kontinuerlig i } D_f$$

$\nexists f'$ då $\frac{2x}{1+x^2} = \pm 1 \Leftrightarrow x = \pm 1$

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}} \cdot \frac{2(1+x^2) - 2x\cdot 2x}{(1+x^2)^2} =$$

$$= \frac{2(1-x^2)}{\frac{\sqrt{1+2x^2+x^4-4x^2}}{\cancel{1+x^2}} \cdot (1+x^2)^{\cancel{2}}} =$$

$$= \frac{2(1-x^2)}{(1+x^2)\underbrace{\sqrt{(1-x^2)^2}}_{=|1-x^2|}} = \begin{cases} \frac{2}{1+x^2}, & 1-x^2>0 \\ -\frac{2}{1+x^2}, & 1-x^2<0 \end{cases}$$

$$= \begin{cases} \frac{2}{1+x^2} & \text{för } |x|<1 \\ -\frac{2}{1+x^2} & \text{för } |x|>1 \end{cases}$$

		-1		1	
f'	$-$		$+$		$-$
f	$\searrow$	$-\frac{\pi}{2}$	$\nearrow$	$\frac{\pi}{2}$	$\searrow$

$\Rightarrow$ f har lok. min i $x=-1$
f har lok. max i $x=1$

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6. $$f(x) = \frac{1}{x-a_1} + \ldots + \frac{1}{x-a_n}$$

D_f: $x \neq a_k$, $k=1\ldots n$

$$f'(x) = -\frac{1}{(x-a_1)^2} - \ldots - \frac{1}{(x-a_n)^2} < 0$$

$\Rightarrow$ f avtagande i alla definitionsintervall

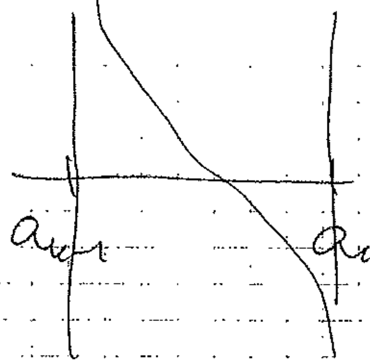
$\Rightarrow$ $\exists$ högst ett nollställe i varje definitionsintervall

$$\lim_{x\to-\infty} f(x) = -0$$

$$\lim_{x\to+\infty} f(x) = +0$$

$$\lim_{x\to a_k-0} f(x) = -\infty \ ; \ \lim_{x\to a_k+0} f(x) = +\infty$$

$k = 1 \dots n$

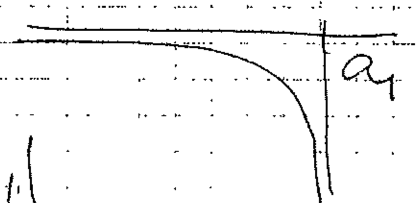


$\Rightarrow \exists\, \varepsilon > 0$:
$f(x) > 0$ i $x = a_{k-1} + \varepsilon$
$\exists\, \delta > 0$:
$f(x) < 0$ i $x = a_k - \delta$

$\Rightarrow \exists$ minst ett nollställe till f i (a_{k-1}, a_k)

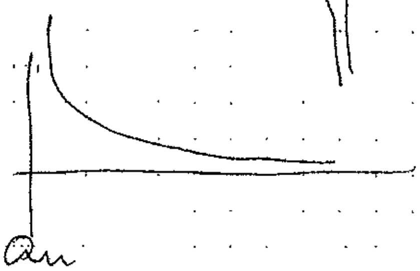
$\Rightarrow \exists$ exakt ett nollställe till f i (a_{k-1}, a_k)

$(-\infty, a_1)$:



f avtagande, < 0 i hela intervallet

$(a_n, +\infty)$:



$f > 0$ i hela intervallet

$\Rightarrow f$ har exakt $\underline{n}-1$ nollställen i $\mathbb{R}$

$c \neq 0$: $c < 0$: $f(x) \overset{\to 0}{-c}$ har $\underline{n}-1$ nollställen, ett i vardera (a_{k-1}, a_k) och ett i $(-\infty, a_1)$; $c > 0$: ett i (a_n, ∞)