TMA 970 Inledande matematisk analys F / TM

Lösningar 30/8 - 2013

1. (a) divergent ; (b) konvergent ;
(c) konvergent ; (d) konvergent ;
(e) falskt ; (f) sant ;
(g) falskt ; (h) sant.

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2. (a) $\sqrt{x^2+4x+4}+x-2 =$
$= |x+2|+x-2 = -x-2+x-2 =$
$= -4$, ty $x \to -\infty$ och
$x+2<0$ utan
inskränkning

$$\Rightarrow \lim_{x\to-\infty}\left(\sqrt{x^2+4x+4}+x-2\right) = \underline{\underline{-4}}$$

(b)
$$\frac{\tan^2 x+\cos x-1}{\sin x^2} =$$

$$= \frac{\frac{\sin^2 x}{\cos^2 x}+\cos x-1}{\sin x^2} =$$

$$= \frac{\sin^2 x}{x^2}\cdot\frac{1}{\cos^2 x}\cdot\frac{x^2}{\sin x^2} - \frac{2\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\cdot\frac{\frac{x^2}{4}}{\sin x^2}$$

$$= \underbrace{\left(\frac{\sin x}{x}\right)^2}_{\to 1,\ x\to 0}\cdot\underbrace{\frac{1}{\cos^2 x}}_{\to 1,\ x\to 0}\cdot\underbrace{\frac{x^2}{\sin x^2}}_{\to 1,\ x\to 0} - \cancel{2}\underbrace{\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2}_{\to 1,\ x\to 0}\cdot\frac{1}{\cancel{4}_{2}}\cdot\underbrace{\frac{x^2}{\sin x^2}}_{\to 1,\ x\to 0} = \underline{\underline{\frac{1}{2}}}$$

(3.) $f(x) = \frac{2x}{x^2+1}$, $D_f = \mathbb{R}$

Nollställe : $x=0$, $f>0$ i $(0,\infty)$

$f(-x) = -f(x) \Rightarrow f$ udda

Asymptoter : inga vertikala

$\lim_{x\to\pm\infty} f(x) = 0 \Rightarrow y=0$ (x-axeln) horisontell asymptot (i $\pm\infty$)

$f'(x) = \frac{2-2x^2}{(x^2+1)^2}$

$f'(x) = 0$: $x = \pm 1$

x		-1		1	
f'	$--$	0	$+$	0	$--$

$\Rightarrow f$ har lok. min i -1 och lok. max i $+1$

(p.g.a. symmetrin räcker det egentligen att titta på $x \geq 0$)

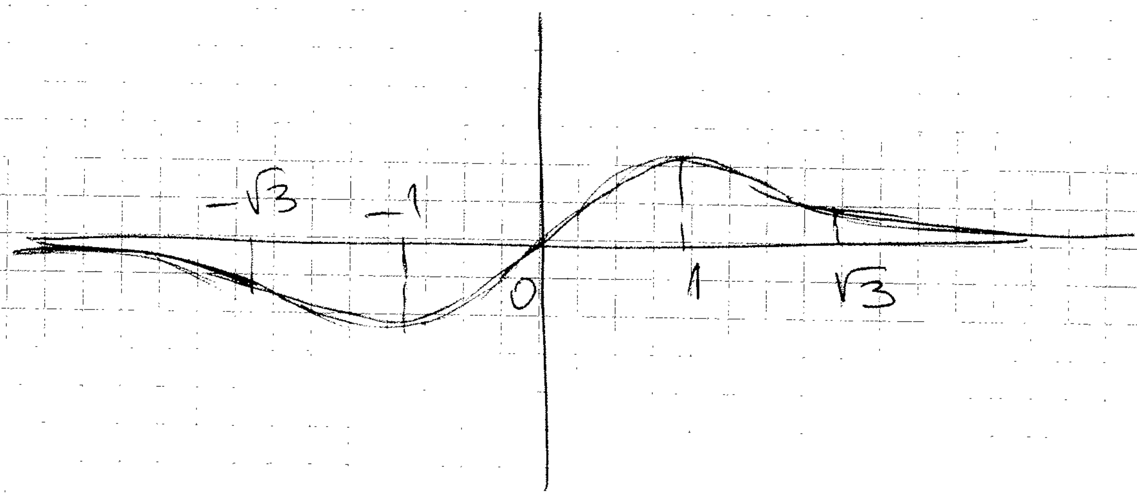
$f''(x) = \frac{-4x(x^2+1)^{\cancel{2}} - 2\cancel{(x^2+1)} \cdot 2x(2-2x^2)}{(x^2+1)^{\cancel{4}\,3}} =$

$= \frac{-4x^3 - 4x - 8x + 8x^3}{(x^2+1)^3} = \frac{4x(x^2-3)}{(x^2+1)^3}$

x		$-\sqrt{3}$		0		$\sqrt{3}$	
f''	$-$	0	$+$	0	$-$	0	$+$
f	konkav		konvex		konkav		konvex

$\Rightarrow f$ har inflexion i $\pm\sqrt{3}$ och 0 $\boxed{3}$

x	$-\infty$		$-\sqrt{3}$	-1	0	1	$\sqrt{3}$		$+\infty$
f	$\to 0$	⌒	infl	lok. min	0	lok. max	infl		$\to 0$
f'		$-$	$-$	0	$+$	0	$-$	$-$	
f''		$-$	0	$+$	0	$-$	0	$+$	



④ (a)
$$\int \frac{dx}{\sqrt{-x^2+6x+3}} = \int \frac{dx}{\sqrt{-(x-3)^2+12}} =$$

$$= \frac{1}{\sqrt{12}} \int \frac{dx}{\sqrt{1-\left(\frac{x-3}{\sqrt{12}}\right)^2}} = \int \frac{\left(\frac{x-3}{\sqrt{12}}\right)' dx}{\sqrt{1-\left(\frac{x-3}{\sqrt{12}}\right)^2}} =$$

$$= \arcsin \frac{x-3}{2\sqrt{3}} \quad (+C)$$

b) $x^3-8 = (x-2)(x^2+2x+4)$

$$\frac{x}{x^3-8} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}$$

$$x = A(x^2+2x+4) + (Bx+C)(x-2)$$

$x=2$: $\quad 2 = 12A \quad \Rightarrow \quad A = 1/6$

x^2 : $\quad 0 = A+B \quad \Rightarrow \quad B = -1/6$

$x = 0 \ (x^0): \quad 0 = 4A - 2C$ (4)

$$\Rightarrow \quad C = \frac{1}{3}$$

$$\Rightarrow \int_0^1 \frac{x}{x^3-8}\,dx = \frac{1}{6}\int_0^1 \frac{dx}{x-2} - \frac{1}{6}\int_0^1 \frac{x-2}{x^2+2x+4}\,dx$$

$$= \frac{1}{6}\Big[\ln|x-2|\Big]_0^1 - \frac{1}{6}\int_0^1 \frac{(x+1)-3}{(x+1)^2+3}\,dx =$$

$$= \frac{1}{6}(0 - \ln 2) - \frac{1}{6}\int_0^1 \frac{x+1}{(x+1)^2+3}\,dx +$$

$$+ \frac{1}{2}\int_0^1 \frac{dx}{(x+1)^2+3} = -\frac{1}{6}\ln 2 -$$

$$- \frac{1}{12}\Big[\ln(x^2+2x+4)\Big]_0^1 + \frac{1}{2\sqrt{3}}\int_0^1 \frac{\frac{1}{\sqrt{3}}\,dx}{1+\left(\frac{x+1}{\sqrt{3}}\right)^2} =$$

$$= \cancel{-\frac{1}{6}\ln 2} - \frac{1}{12}\ln 7 + \cancel{\frac{1}{12}\ln 4} +$$

$$+ \frac{1}{2\sqrt{3}}\left[\arctan\frac{x+1}{\sqrt{3}}\right]_0^1 =$$

$$= -\frac{1}{12}\ln 7 + \frac{1}{2\sqrt{3}}\arctan\frac{2\sqrt{3}}{3} - \frac{1}{2\sqrt{3}}\cdot\frac{\pi}{6}$$

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5. $a_n > 0 \quad \forall n \in \mathbb{N}$

$$a_{n+1} = a_n \cdot \underbrace{\left(1 - \frac{1}{2^{n+1}}\right)}_{<1} < a_n$$

$\Rightarrow$ följden är avtagande och nedåt begränsad

$$\Rightarrow \exists \lim_{n\to\infty} a_n$$

6. $f'' > 0$ i (a,b) 5

$\Rightarrow f'$ strängt växande i (a,b)

För $x_1 \neq x_2$

$$\frac{f(x_1)+f(x_2)}{2} - f\left(\frac{x_1+x_2}{2}\right) =$$

$$= \frac{f(x_2) - f\left(\frac{x_1+x_2}{2}\right)}{2} - \frac{f\left(\frac{x_1+x_2}{2}\right) - f(x_1)}{2} =$$

$$= \frac{1}{2} f'(\xi)\left(x_2 - \frac{x_1+x_2}{2}\right) - \frac{1}{2} f'(\eta)\left(\frac{x_1+x_2}{2} - x_1\right) =$$

$$= \frac{1}{4}(x_2 - x_1)\left(f'(\xi) - f'(\eta)\right),$$

enligt medelvärdessatsen, där

$$\underbrace{\xi \in \left(\frac{x_1+x_2}{2}, x_2\right), \quad \eta \in \left(x_1, \frac{x_1+x_2}{2}\right)}$$

$\Rightarrow \xi > \eta$

$\Rightarrow f'(\xi) > f'(\eta)$

$$\Rightarrow \frac{f(x_1)+f(x_2)}{2} - f\left(\frac{x_1+x_2}{2}\right) \geq 0$$

(om $x_1 = x_2$: "=")