TMA970 Inledande matematisk analys F/TM

Lösningar 25/8-17

1. (a) konvergent; (b) konvergent;
(c) divergent; (d) konvergent;
(e) konvergent; (f) konvergent.

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2. (a)
$$\frac{\ln(x^2-x+1)}{\ln(x^{10}+x+1)} = \frac{\ln\left(x^2\left(1-\frac{1}{x}+\frac{1}{x^2}\right)\right)}{\ln\left(x^{10}\left(1+\frac{1}{x^9}+\frac{1}{x^{10}}\right)\right)} =$$
$$= \frac{\ln x^2 + \ln\left(1-\frac{1}{x}+\frac{1}{x^2}\right)}{\ln x^{10} + \ln\left(1+\frac{1}{x^9}+\frac{1}{x^{10}}\right)} =$$
$$= \frac{2\ln x + \ln\left(1-\frac{1}{x}+\frac{1}{x^2}\right)}{10\ln x + \ln\left(1+\frac{1}{x^9}+\frac{1}{x^{10}}\right)} =$$
$$= \frac{2 + \ln\overbrace{\left(1-\frac{1}{x}+\frac{1}{x^2}\right)}^{\to 1}/\overbrace{\ln x}^{\to\infty}}{10 + \ln\underbrace{\left(1+\frac{1}{x^9}+\frac{1}{x^{10}}\right)}_{\to 1}/\underbrace{\ln x}_{\to\infty}} \longrightarrow \frac{2+0}{10+0} = \underline{\underline{\frac{1}{5}}}$$

(b) Sätt $1-x=t \Rightarrow x = 1-t$
$$\tan\frac{\pi x}{2} = \tan\frac{\pi(1-t)}{2} = \frac{\sin\left(\frac{\pi}{2}-\frac{\pi t}{2}\right)}{\cos\left(\frac{\pi}{2}-\frac{\pi t}{2}\right)} =$$
$$= \frac{\cos\frac{\pi t}{2}}{\sin\frac{\pi t}{2}}\,; \qquad x\to 1 \Leftrightarrow t\to 0$$

$$t \cdot \frac{\cos\left(\frac{\pi}{2}t\right)}{\sin\left(\frac{\pi}{2}t\right)} = \frac{t\frac{\pi}{2}}{\sin\left(\frac{\pi}{2}t\right)} \cdot \cos\left(\frac{\pi}{2}t\right) \cdot \frac{2}{\pi}$$

$$\xrightarrow[t\to 0]{} 1 \cdot 1 \cdot \frac{2}{\pi}$$

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3. D_f : $x^2+3>0 \quad \forall x \in \mathbb{R}$

$\Rightarrow D_f = \mathbb{R}$

$x^2+3 \geq 3 \quad \forall x \in \mathbb{R} \Rightarrow f(x) \geq \ln 3 \quad \forall x \in \mathbb{R}$ min > 1

(ty ln växande)

$f(-x) = f(x) \Rightarrow f$ jämn (räcker att betrakta $[0,\infty)$)

ej periodisk

$\lim_{x\to\pm\infty} f(x) = +\infty$

$\frac{f(x)}{x} = \frac{\ln(x^2+3)}{x} \xrightarrow[x\to\pm\infty]{} 0$, men $f(x) \xrightarrow[\pm\infty]{} \infty$

$\Rightarrow$ inga asymptoter i $\pm\infty$

$f'(x) = \frac{1}{x^2+3} \cdot 2x = 0$ för $x=0$

$f'<0$ för $x<0$; $f'>0$ för $x>0$

$\Rightarrow f$ har min (lok och globalt) i 0

(vi hittade det ovan)

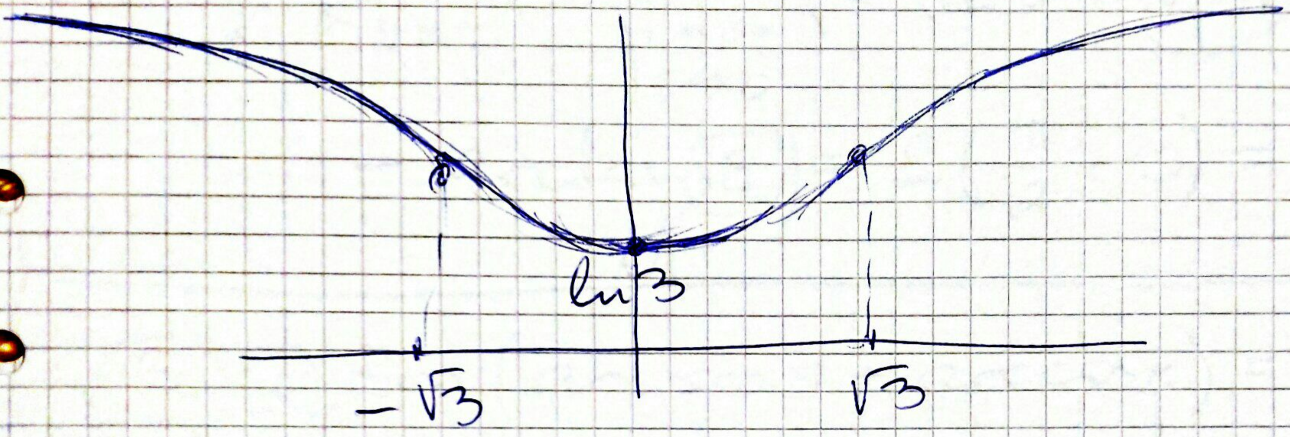
$$f''(x) = \frac{2(x^2+3)-4x^2}{(x^2+3)^2} = 2 \cdot \frac{3-x^2}{(x^2+3)^2}$$

$f''=0$ för $x = \pm\sqrt{3}$; $f''<0$ för $|x|>\sqrt{3}$; $f''>0$ för $|x|<\sqrt{3}$

$\Rightarrow f$ konvex i $(-\sqrt{3}, \sqrt{3})$, konkav annars

inflexion i $\pm\sqrt{3}$

x	$-\infty$		$-\sqrt{3}$		0		$\sqrt{3}$		$+\infty$
f	∞	↘		↘	$\ln 3$	↗		↗	∞
f'		$-$	$-$		0	$+$		$+$	
f''		$-$	0	$+$		$+$	0	$-$	



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4. (a) $\displaystyle\int \frac{dx}{(\cos x+\sin x)^2} = \int \frac{dx}{\cos^2 x\,(1+\tan x)^2} =$

$$= \int \frac{(\tan x)'\,dx}{(1+\tan x)^2} = \frac{-1}{1+\tan x} \quad (+C)$$

Alternativt: $\displaystyle\int \frac{dx}{(\cos x+\sin x)^2} = \int \frac{dx}{1+\sin 2x}$,

sedan standardsubstitutionen.

(b) $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{1+x}} = \lim_{\substack{\varepsilon\to 0\\ p\to\infty}} \int_{1+\varepsilon}^{p} \frac{dx}{x\sqrt{1+x}}$

Primitiv funktion: Sätt $t=\sqrt{1+x}$

$\Rightarrow x = 1+t^2, \quad dx = +2t\,dt$

$\boxed{4}$

$$\int \frac{dx}{x\sqrt{x-1}} = \int \frac{2\cancel{t}\,dt}{(1+t^2)\cdot \cancel{t}} =$$

$$= 2\arctan t + C = 2\arctan\sqrt{x-1} + C$$

$$\Rightarrow \int_1^{\infty} \frac{dx}{x\sqrt{x-1}} = \lim_{\substack{p\to\infty \\ \varepsilon\to 0}} \Big(2\big(\arctan\underbrace{\sqrt{p-1}}_{\to\infty} -$$

$$- \arctan\underbrace{\sqrt{\varepsilon}}_{0}\big)\Big) = 2\left(\frac{\pi}{2} - 0\right) = \pi$$

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5. (i) $\sin(\arcsin x + \arccos x) =$

$$= \sin(\arcsin x)\cos(\arccos x) +$$
$$+ \cos(\arcsin x)\sin(\arccos x) =$$
$$= x\cdot x + \sqrt{1-\sin^2(\arcsin x)}\sqrt{1-\cos^2(\arccos x)} \quad (*)$$
$$= x^2 + \sqrt{1-x^2}\cdot\sqrt{1-x^2} = x^2 + 1 - x^2 = 1$$

$(*)$: båda $\sqrt{}$ med $+$, eftersom

$\arcsin x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, där $\cos \geq 0$

$\arccos x \in [0, \pi]$, där $\sin \geq 0$

$\arcsin x + \arccos x \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

sin är endast $= 1$ för vinkel $\frac{\pi}{2}$ i det intervallet

$$\Rightarrow \arcsin x + \arccos x = \frac{\pi}{2}$$

(ii) $f(x) = \arcsin x + \arccos x$

$f'(x) \equiv 0 \quad \forall x \in [-1,1] \Rightarrow f \equiv C$ i $[-1,1]$

$f(0) = 0 + \pi/2 \Rightarrow C = \pi/2$

6. $\left(\frac{1+\sqrt[n]{a}}{2}\right)^n = e^{n\ln\frac{1+\sqrt[n]{a}}{2}} =$

$= e^{n\ln\left(1-\underbrace{\frac{1-\sqrt[n]{a}}{2}}_{\to 0,\ n\to\infty}\right)} = e^{n\underbrace{\frac{\ln\left(1-\frac{1-\sqrt[n]{a}}{2}\right)}{-(1-\sqrt[n]{a})/2}}_{\downarrow n\to\infty\ \ 1} \cdot \frac{-1+\sqrt[n]{a}}{2}}$

standardlim

$n\frac{-1+\sqrt[n]{a}}{2} = \frac{1}{2}n\left(e^{\frac{1}{n}\ln a}-1\right) =$

$= \frac{1}{2}\underbrace{\frac{e^{\frac{1}{n}\ln a \to 0}-1}{\frac{1}{n}\ln a}}_{\downarrow n\to\infty\ \ 1} \cdot \ln a \to \frac{1}{2}\ln a$

standardlim

$\Rightarrow$ det sökta gränsvärdet är

$$e^{1\cdot\frac{1}{2}\ln a} = e^{\ln\sqrt{a}} = \underline{\underline{\sqrt{a}}}$$

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8. se lösningen till tidigare tenta, augusti 2016