TMA970 Inledande matematisk analys F/TM

Lösningar 5/1-2015

1. (a) konvergent; (b) konvergent;
(c) konvergent; (d) falskt;
(e) sant; (f) sant.

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2. (a) $\ln(2x+1)-\ln(x+1) = \ln\frac{2x+1}{x+1} =$

$$= \ln\frac{2+1/x}{1+1/x} \xrightarrow[x\to\infty]{} \ln 2$$

(b) $\frac{1}{x}\ln\sqrt{\frac{1+x}{1-x}} = \frac{1}{x}\cdot\frac{1}{2}\left(\ln(1+x)-\ln(1-x)\right) =$

$$= \frac{1}{2}\,\frac{\ln(1+x)}{x} + \frac{1}{2}\,\frac{\ln(1-x)}{-x} \xrightarrow[x\to 0]{} \frac{1}{2}+\frac{1}{2} = 1$$

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3. $f(x) = \sqrt{x}+\sqrt{4-x}$

D_f : $x \geq 0$ och $4-x\geq 0$

$\Rightarrow D_f = [0,4]$

$\lim_{x\to 0} f(x) = f(0) = 2$

$\lim_{x\to 4} f(x) = f(4) = 2$

Inga asymptoter.

$f(x) = \underbrace{\sqrt{\ldots}}_{\geq 0} + \underbrace{\sqrt{\ldots}}_{\geq 0} \geq 0 \quad \forall\, x\in D_f$

$f(x)=0$ om båda termerna $=0$ samtidigt

(2)

$\sqrt{x}$ och $\sqrt{4-x}$ kan inte vara $=0$ samtidigt

$\Rightarrow f(x) > 0 \quad \forall x \in [0,4]$

ej jämn/udda, ej periodisk (det finns dock en symmetri, vilket man inser om man gör variabelbytet $x = t+2$)

$$f'(x) = \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{4-x}} \cdot (-1) =$$

$$= \frac{\sqrt{4-x} - \sqrt{x}}{2\sqrt{x}\sqrt{4-x}} = 0 \quad \text{för } \overset{\geq 0}{\sqrt{x}} = \overset{\geq 0}{\sqrt{4-x}} \;\Big|^2$$

$\Leftrightarrow x = 4-x$

$\Leftrightarrow x = 2$

$f(2) = 2\sqrt{2}$

x	0		2		4
f'	$+\infty$	+	0	−	$-\infty$

$$f''(x) = -\frac{1}{4} \cdot \frac{1}{\sqrt{x^3}} - \left(-\frac{1}{4}\right) \frac{1}{\sqrt{(4-x)^3}} \cdot (-1) < 0 \quad \forall x \in (0,4)$$

$D_{f'} = D_{f''} = (0,4)$

$\Rightarrow f$ konkav i D_f

x	0		2		4
f	2	$\nearrow$	$2\sqrt{2}$	$\searrow$	2
f'	$+\infty$	+	0	−	$-\infty$
f''	−	−	−	−	−

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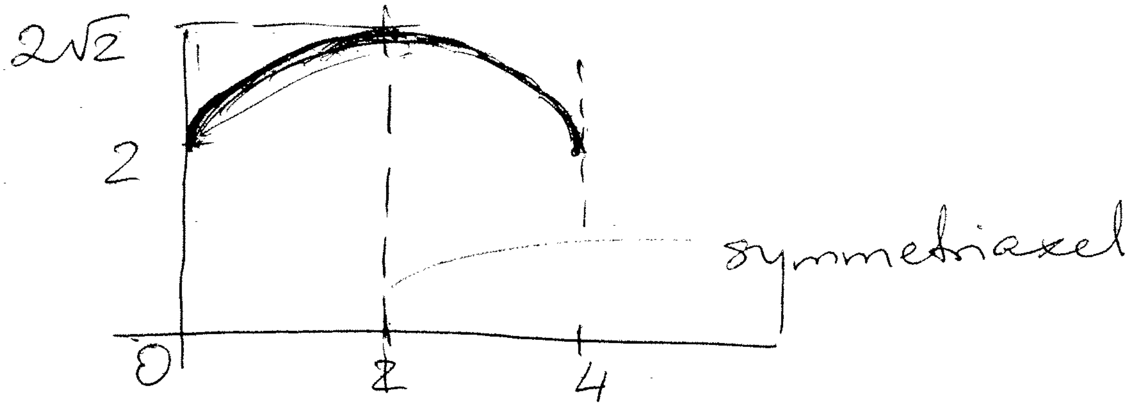

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4. (a) $$\int \frac{x^2}{\sqrt{x^2+9}}\,dx = \frac{1}{2}\int x \cdot \frac{(x^2+9)'}{\sqrt{x^2+9}}\,dx =$$

$$= \frac{1}{2}\int x \cdot 2\left(\sqrt{x^2+9}\right)'dx =$$

$$= x\sqrt{x^2+9} - \int \sqrt{x^2+9}\,dx$$

$$\int \frac{x^2}{\sqrt{x^2+9}}\,dx = \int \frac{x^2+9-9}{\sqrt{x^2+9}}\,dx =$$

$$= \int \sqrt{x^2+9}\,dx - 9\int \frac{dx}{\sqrt{x^2+9}}$$

$$\Rightarrow \quad x\sqrt{x^2+9} - \int\sqrt{x^2+9}\,dx = \int\sqrt{x^2+9}\,dx - 9\int\frac{dx}{\sqrt{x^2+9}}$$

$$\Rightarrow 2\int\sqrt{x^2+9}\,dx = x\sqrt{x^2+9} + 9\ln\left(x+\sqrt{x^2+9}\right) \quad (+C)$$

$$\Rightarrow \int\frac{x^2}{\sqrt{x^2+9}}\,dx = \frac{1}{2}x\sqrt{x^2+9} - \frac{9}{2}\ln\left(x+\sqrt{x^2+9}\right)$$

(Kan även angripas med substitutionerna $x = 3\tan t$ och $x+\sqrt{x^2+9} = t$.)

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(b) $\int_0^{\frac{\pi}{4}} \frac{dx}{1+3\cos^2 x} = \int_0^{\frac{\pi}{4}} \frac{dx}{\sin^2 x + 4\cos^2 x} =$

$= \int_0^{\frac{\pi}{4}} \frac{\frac{1}{\cos^2 x}\, dx}{\tan^2 x + 4} = \left[\begin{array}{l} t = \tan x \\ x=0 \,;\; t=0 \\ x=\frac{\pi}{4} \,;\; t=1 \,, \\ dt = \frac{1}{\cos^2 x}\, dx \end{array}\right] =$

$= \int_0^1 \frac{dt}{t^2+4} = \cancel{2}\frac{1}{\cancel{4}_2} \int_0^1 \frac{\frac{1}{2}dt}{1+\left(\frac{t}{2}\right)^2} =$

$= \frac{1}{2}\left[\arctan \frac{t}{2}\right]_0^1 = \frac{1}{2}\arctan\frac{1}{2}$

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5. $f(x) = 2x + 3\sqrt[3]{x^2}$, $D_f = \mathbb{R}$

$f'(x) = 2 + \cancel{3}\cdot\frac{1}{\cancel{3}}\cdot(x^2)^{-\frac{2}{3}}\cdot 2x =$

$= 2\left(1+\frac{1}{\sqrt[3]{x}}\right)$ $\qquad D_{f'} : x \neq 0$

$\Rightarrow$ lokala extrema kan finnas där $f'=0$ eller där $\nexists f'$

$2+\frac{1\cdot 2}{\sqrt[3]{x}} = 0 \quad \vdots \quad \cancel{2}\sqrt[3]{x} = -1\cdot\cancel{2} \Big|^3$

$\Leftrightarrow x = -1$

$\nexists f' : x = 0$

x		-1		0	
f'	$+$	0	$-$	$\cancel{\infty}\,\nexists$	$+$

$\Rightarrow f$ har lok. max i -1 och lok. min i 0

(6.) $$\int_0^{\frac{\pi}{2}} \cos^n x \, dx = \left[\begin{array}{l} x = \frac{\pi}{2} - t \\ dx = -dt \\ x=0 : t = \frac{\pi}{2} ; x = \frac{\pi}{2} : t=0 \end{array}\right]$$ (5)

$$= \int_{\frac{\pi}{2}}^{0} \underbrace{\left(\cos\left(\frac{\pi}{2} - t\right)\right)^n}_{= \sin t} (-1) \, dt =$$

$$= \int_0^{\frac{\pi}{2}} \sin^n t \, dt = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$$

Alternativ lösning: m.h.a. induktion

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx ; \quad J_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$$

(1) $I_k = J_k$ kollas direkt för $k=1,2$

(2) Antag likheten sann för $n = m$, för ngt $m \in \mathbb{N}$

(3) Är den då sann för $n = m+1$?

$$I_{m+1} = \int_0^{\frac{\pi}{2}} \cos^m x \underbrace{\cos x}_{=(\sin x)'} \, dx = \underbrace{\left[\sin x \cos^m x\right]_0^{\frac{\pi}{2}}}_{=0} -$$

$$- \int_0^{\frac{\pi}{2}} \sin x \cdot m \cos^{m-1} x \cdot (-\sin x) \, dx =$$

$$= \int_0^{\frac{\pi}{2}} (1 - \cos^2 x) \cdot m \cdot \cos^{m-1} x \, dx =$$

$$= m I_{m-1} - m I_{m+1} \text{, och likadant}$$

$\Rightarrow$ följer likheten för J_{m+1}