TMA970 Inledande matematisk analys F1 / TM1

Lösningar 29/10-2015

1. (a) konvergent ; (b) konvergent ;
(c) konvergent ; (d) sant ;
(e) sant ; (f) sant.

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2. (a)
$$\frac{\tan x - \sin x}{x^3} = \frac{\sin x}{x} \cdot \frac{\frac{1}{\cos x} - 1}{x^2} =$$
$$= \frac{\sin x}{x} \cdot \frac{1-\cos x}{x^2} \cdot \frac{1}{\cos x} =$$
$$= \frac{\sin x}{x} \cdot \frac{1-\cos^2 x}{x^2} \cdot \frac{1}{1+\cos x} \cdot \frac{1}{\cos x} =$$
$$= \left(\frac{\sin x}{x}\right)^3 \cdot \frac{1}{1+\cos x} \cdot \frac{1}{\cos x} \xrightarrow[x\to 0]{} 1 \cdot \frac{1}{2} \cdot 1 = \underline{\underline{\frac{1}{2}}}$$

(b)
$$\frac{\sqrt{1+6x}-5}{\sqrt{x}-2} = \frac{1+6x-25}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{1+6x}+5}$$
$$= \frac{6\cancel{(x-4)}}{\cancel{x-4}} \cdot \frac{\sqrt{x}+2}{\sqrt{1+6x}+5} \xrightarrow[x\to 2^2]{} 6 \cdot \frac{2+2}{\sqrt{25}+5} = \underline{\underline{\frac{12}{5}}}$$

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3. $f(x) = x^2 e^{\frac{1}{x}}$ $\quad D_f = \mathbb{R} \setminus \{0\}$

$x > 0 \quad \forall x \in D_f$

$$\lim_{x\to 0_-} f(x) = \text{"} +0 \cdot e^{-\infty} \text{"} = +0$$

$$\lim_{x\to 0_+} f(x) = \lim_{x\to 0_+} \frac{1}{1/x^2} e^{\frac{1}{x}} = \lim_{t\to +\infty} \frac{e^t}{t^2} = \infty$$

$\Rightarrow x=0$ är vertikal asymptot

$\lim_{x\to\pm\infty} f(x) = "+\infty \cdot e^{0}" = \infty$

$\frac{f(x)}{x} = xe^{\frac{1}{x}} \xrightarrow[x\to\pm\infty]{} \pm\infty$

$\Rightarrow \nexists$ sneda asymptoter

f ej jämn/udda, ej periodisk

$f'(x) = 2xe^{\frac{1}{x}} + \cancel{x^2}\cdot e^{\frac{1}{x}}\cdot\left(-\frac{1}{\cancel{x^2}}\right) =$

$= e^{\frac{1}{x}}(2x-1)$

$f'(x)=0 \iff x=\frac{1}{2}$

x		0		$\frac{1}{2}$	
f'	$-$	|	$-$	0	$+$

$\Rightarrow f$ har lok. min i $\frac{1}{2}$

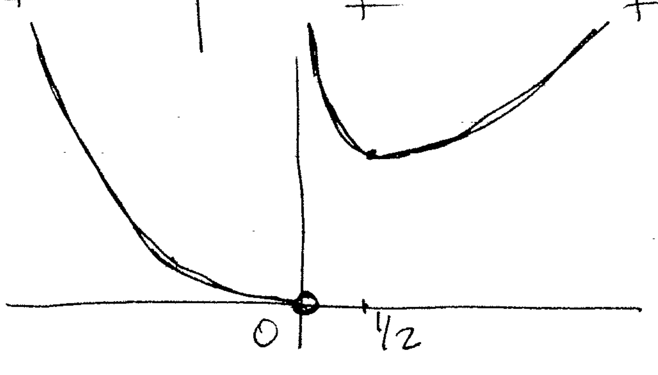
$f(\frac{1}{2}) = \frac{1}{4}e^{2}$ (knappt 2)

$y=0$ "tangent" i origo från vänster
(origo $\notin$ grafen)

$f''(x) = e^{\frac{1}{x}}\frac{2x^2-2x+1}{x^2} > 0$ i D_f

$\Rightarrow f$ konvex i $(-\infty,0)$ och i $(0,\infty)$

x		0		$\frac{1}{2}$	
f	$+\infty \searrow$	$+0$ | $+\infty$	$\searrow$	lok.min	$\nearrow$ $+\infty$
f'	$-$	$+0$ |	$-$	0	$+$
f''	$+$	|	$+$		$+$



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4. (a) $\int \frac{dx}{\sqrt{\sqrt{x}-2}} = \left[\begin{array}{l} \sqrt{x}-2 = t^2 \\ x = (t^2+2)^2 \\ dx = 2(t^2+2)\cdot 2t\,dt \end{array} \right]$

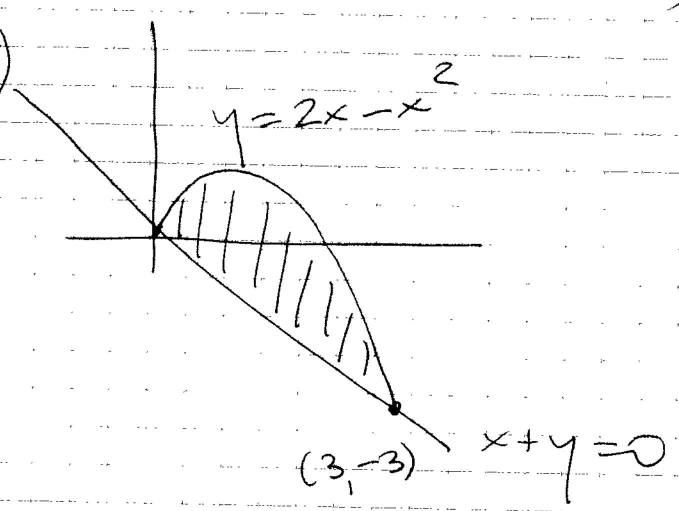
$= \int \frac{4t^3+8t}{t}\,dt = 4\int (t^2+2)\,dt =$

$= \frac{4}{3}t^3 + 8t + C = \frac{4}{3}\left(\sqrt{\sqrt{x}-2}\right)^3 +$

$+ 8\sqrt{\sqrt{x}-2} + C$

(Alternativt: $\sqrt{x} = u$; $\sqrt{u-2} = v \ldots$)

(b)



$2x - x^2 = -x \iff x^2 - 3x = 0$

$\iff x(x-3) = 0 \iff \left| \begin{array}{l} x=0 \\ \text{eller} \\ x=3 \end{array} \right.$

$\Rightarrow$ skärningspunkter $(0,0)$ och $(3,-3)$

$2x - x^2 > -x$ i $(0,3)$

Arean $= \int_0^3 (2x - x^2 - (-x))\,dx =$

$= \left[x^2 - \frac{x^3}{3} + \frac{x^2}{2} \right]_0^3 = \underline{\underline{\frac{9}{2}}}$

5. $f(x) = \arcsin(\cos x)$; $D_f = \mathbb{R}$

$\cos x$ jämn, 2π-periodisk

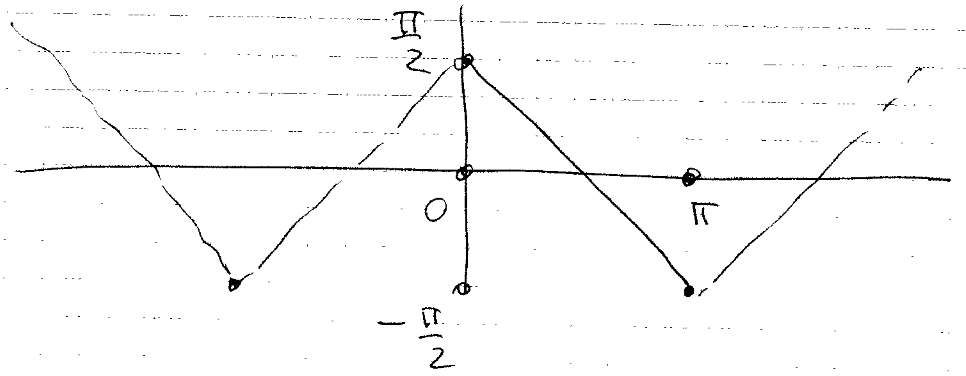
$\Rightarrow$ $f(x)$ jämn, 2π-periodisk

$\Rightarrow$ räcker att rita grafen i $[0,\pi]$

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

$$\Rightarrow f(x) = \arcsin\left(\sin\left(\frac{\pi}{2} - x\right)\right) = \frac{\pi}{2} - x$$

för $\frac{\pi}{2} - x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Leftrightarrow x \in [0,\pi]$



etc

6. $a_n > 0 \ \forall n \in \mathbb{N}$

$\{a_n\}_1^\infty$ växande $\Leftrightarrow \frac{a_{n+1}}{a_n} \geq 1 \ \forall n \in \mathbb{N}$

avtagande $\Leftrightarrow \frac{a_{n+1}}{a_n} \leq 1$ —"—

$$\Leftrightarrow \frac{a_n}{a_{n+1}} \geq 1 \ \forall n \in \mathbb{N}$$

$$\frac{a_n}{a_{n+1}} = \frac{\left(\frac{n+1}{n}\right)^{n+1}}{\left(\frac{n+2}{n+1}\right)^{n+2}} = \frac{n+1}{n+2} \cdot \left(\frac{(n+1)^2}{n(n+2)}\right)^{n+1} =$$

$$= \frac{n+1}{n+2} \cdot \Big(1 + \underbrace{\frac{1}{n(n+2)}}_{= x > -1}\Big)^{n+1} \underset{\text{Bernoullis olikhet}}{\geq} \frac{n+1}{n+2}\left(1 + \frac{n+1}{n(n+2)}\right) =$$

$$= \frac{n+1}{n+2} \cdot \frac{n^2+2n+n+1}{n(n+2)} =$$

$$= \frac{n^3+3n^2+n+n^2+3n+1}{n(n+2)^2} =$$

$$= \frac{n^3+4n^2+4n+1}{n^3+4n^2+4n} > 1$$

$$\Rightarrow \frac{a_n}{a_{n+1}} \geq 1 \quad \forall n \in \mathbb{N}$$

$$\Rightarrow a_{n+1} \leq a_n \quad \forall n \in \mathbb{N}$$

$\Rightarrow$ följden avtagande

$$\lim_{n\to\infty} a_n \lim_{n\to\infty}\left(1+\frac{1}{n}\right) \cdot \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1 \cdot e = e$$