TMA970 Inledande matematisk analys F1/TM1

Lösningar 27/10-16

1. (a) divergent; (b) konvergent;
(c) konvergent; (d) sant;
(e) falskt; (f) falskt.

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2. (a)
$$\frac{\ln \sin 2x}{\ln \sin x} = \frac{\ln(2\sin x\cos x)}{\ln \sin x} =$$

$$= \frac{\ln 2 + \ln \sin x + \ln \cos x}{\ln \sin x} =$$

$$= \frac{\ln 2 + \overbrace{\ln \cos x}^{\xrightarrow[x\to 0_+]{} 0}}{\underbrace{\ln \sin x}_{\xrightarrow[x\to 0_+]{} -\infty}} + 1 \xrightarrow[x\to 0_+]{} 1$$

(b)
$$\frac{\sqrt[3]{7+x^3} - \sqrt{3+x^2}}{x-1} =$$

$$= \frac{x\sqrt[3]{1+\frac{7}{x^3}} - |x|\sqrt{1+\frac{3}{x^2}}}{x\left(1-\frac{1}{x}\right)} = \left[\begin{array}{l} x \to -\infty \\ \Rightarrow x < 0 \\ |x| = -x \end{array}\right]$$

$$= \frac{\cancel{x}\left(\sqrt[3]{1+\frac{7}{x^3}} + \sqrt{1+\frac{3}{x^2}}\right)}{\cancel{x}\left(1-\frac{1}{x}\right)} \xrightarrow[x\to -\infty]{} \frac{1+1}{1} = 2$$

3. $f(x) = \frac{x}{\sqrt{x-1}}$

D_f : $x > 1$

$f(x) > 0 \quad \forall x \in D_f$

$\lim_{x\to 1_+} f(x) = "\frac{1}{+0}" = +\infty$

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\cancel{\sqrt{x}}\cdot\sqrt{x}}{\cancel{\sqrt{x}}\sqrt{1-\frac{1}{x}}} = +\infty$

$\left.\begin{array}{l} \frac{f(x)}{x} = \frac{1}{\sqrt{x-1}} \xrightarrow[x\to\infty]{} 0 \\ f(x) - 0\cdot x \xrightarrow[x\to\infty]{} \infty \end{array}\right\} \Rightarrow \nexists$ sned asymptot i ∞

$x = 1$ vertikal asymptot

mga symmetrier

$f'(x) = \frac{\sqrt{x-1} - \frac{1}{2\sqrt{x-1}}\cdot x}{x-1} =$

$= \frac{2(x-1) - x}{2(x-1)^{3/2}} = \frac{x-2}{2(x-1)^{3/2}}$

x	1		2		∞
f'		$-$	0	$+$	

$\Rightarrow f$ har lok. min i $x_0 = 2$

$f''(x) = \frac{2(x-1)^{3/2} - \cancel{2}\cdot\frac{3}{\cancel{2}}(x-1)^{1/2}\cdot(x-2)}{2(x-1)^3} =$

$= \frac{2x-2-3x+6}{2(x-1)^{5/2}} = \frac{-x+4}{2(x-1)^{5/2}}$

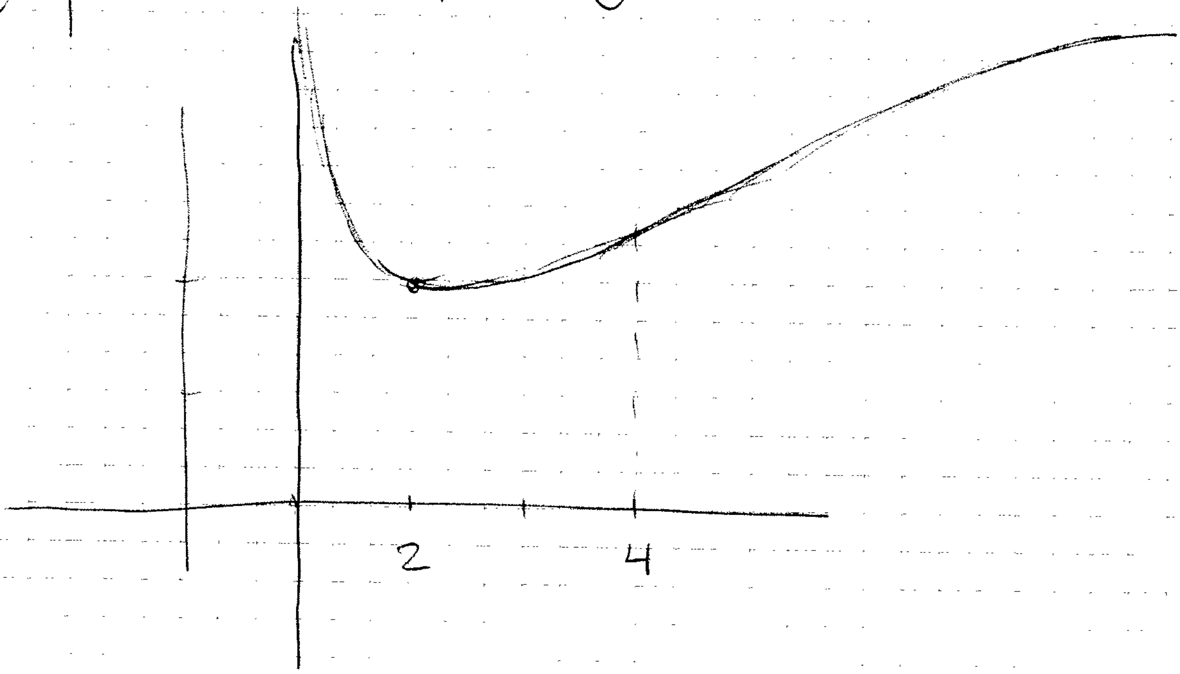
x	1		4		∞
f''		$+$	0	$-$	

$\Rightarrow$ f konvex i $(1, 4)$

konkav i $(4, \infty)$

f har inflexionspkt för $x = 4$

x	1		2		4		∞
f	$+\infty$	$\searrow$	2 lok. min	$\nearrow$	infl.	$\nearrow$	$+\infty$
f'		$-$	0	$+$		$+$	
f''		$+$	$+$	$+$	0	$-$	$-$



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4. (a) $\dfrac{x+1}{(x^2-2x+4)(x+3)^2} =$

$$= \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{Cx+D}{x^2-2x+4} \quad \Big|\ \cdot Q(x)$$

$$x+1 = A(x+3)(x^2-2x+4) + B(x^2-2x+4) +$$
$$+ (Cx+D)(x+3)^2$$

$x = -3: \quad -2 = 19B \Rightarrow B = -2/19$

$x^3: \quad 0 = A + C \Rightarrow A = -C$

x^0 $(x=0)$: $\quad 1 = 12A + 4B + 9D$

$x^2: \quad 0 = A - 2B + 6C + D =$

$= -2B + 5C + D =$

$= -5A - 2B + D$

$$\begin{cases} 12A + 9D = 1 + 8/19 = 27/19 \\ -5A + D = 2B = -4/19 \quad \cdot(-9) \end{cases}$$

$57A = \frac{36+27}{19} = \frac{63}{19}$

$19A = \frac{21}{19} \qquad A = -C = \frac{21}{361}$

$D = 5A - \frac{4}{19} = \frac{105-76}{361} = \frac{29}{361}$

(i) $\int \frac{A}{x+3} = A\ln|x+3|$ (vi väljer alla konstanter $=0$)

(ii) $\int \frac{B}{(x+3)^2} = -B \cdot \frac{1}{x+3}$

(iii) $\int \frac{Cx+D}{x^2-2x+4}\,dx = \int \frac{C(x-1)+(C+D)}{(x-1)^2+3} =$

$= \frac{C}{2}\ln(x^2-2x+4) + (C+D)\cdot\frac{1}{3}\int \frac{dx}{\left(\frac{x-1}{\sqrt{3}}\right)^2+1} =$

$= \frac{C}{2}\ln(x^2-2x+4) + (C+D)\cdot\frac{\sqrt{3}}{3}\arctan\frac{x-1}{\sqrt{3}}$

Återstår att lägga ihop resultaten från (i), (ii), (iii)!

(b) $\int_1^2 \frac{dx}{\sqrt{x}+\sqrt[3]{x}} = \left[\begin{array}{l} x = t^6 \\ dx = 6t^5\,dt \\ x=1: t=1;\ x=2: t=\sqrt[6]{2} \end{array}\right.$ (5)

$= \int_1^{\sqrt[6]{2}} \frac{6t^{\cancel{5}3}}{\cancel{t^3}^{1}\ t^{2}\cancel{}+t^{\cancel{3}}}\,dt$

$$= \int_1^{\sqrt[6]{2}} \frac{6t^3}{1+t}\,dt = 6\int_1^{\sqrt[6]{2}} \frac{(t^3+1)-1}{t+1}\,dt =$$

$$= 6\int_1^{\sqrt[6]{2}} \left(t^2-t+1-\frac{1}{t+1}\right)dt =$$

$$= \left[2t^3-3t^2+6t-6\ln(t+1)\right]_1^{\sqrt[6]{2}} =$$

$$= 2\sqrt{2}-3\sqrt[3]{2}+6\sqrt[6]{2}-6\ln(1+\sqrt[6]{2}) -$$

$$\underbrace{-2+3-6}_{=-5}+6\ln 2$$

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5. $\cos \text{v.l.} = \sqrt{1-x^2} =$ | $D_f: |x|\le 1$

$= \cos \text{h.l.} = \cos(-\arcsin x) =$

$= \cos(\arcsin x) = \sqrt{1-x^2}$

ty cos jämn

ty $\arcsin \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\cos \ge 0$ i $\uparrow$

v.l. $\in [0,\pi]$ h.l. $\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

För att de ska ligga i samma intervall ska båda ligga i $\left[0, \frac{\pi}{2}\right]$

$\sqrt{1-x^2} \ge 0$

$\Rightarrow \arccos\sqrt{1-x^2} \ge 0$ (alltid)

$\quad$ — " — $\le \frac{\pi}{2}$

$-\arcsin x \in \left[0, \frac{\pi}{2}\right]$

$\Leftrightarrow \arcsin x \le 0$

$\Leftrightarrow \boxed{x \le 0}$ ← Svar

6. (a) $\dfrac{f(x)-f(0)}{x-0} = \dfrac{\frac{\sin x}{x} - 1}{x}$

6

$x > 0$: $\quad \cos x < \dfrac{\sin x}{x} < 1$

$\Rightarrow \quad \cos x - 1 < \dfrac{\sin x}{x} - 1 < 0$

$\Rightarrow \quad \dfrac{\cos x - 1}{x} < \dfrac{f(x)-f(0)}{x-0} < 0$

$= \dfrac{(\cos x - 1)(\cos x + 1)}{x(\cos x + 1)} = \dfrac{\cos^2 x - 1}{x(\cos x + 1)} =$

$= \dfrac{-\sin^2 x}{x(\cos x + 1)} = -\dfrac{\sin^2 x}{x^2}\dfrac{1}{(\cos x + 1)}\, x$

$\dfrac{\sin^2 x}{x^2} \xrightarrow{x\to 0} 1$, $\quad \dfrac{1}{\cos x + 1} \xrightarrow{x \to 0} \dfrac{1}{2}$, $\quad x \xrightarrow{x\to 0} 0$

$\Rightarrow \quad \dfrac{\cos x - 1}{x} \xrightarrow[x\to 0]{} 0$

$\Rightarrow \quad \dfrac{f(x)-f(0)}{x-0} \xrightarrow[x\to 0]{} 0$

enligt instängningsregeln

$\Rightarrow$ f har högerderivat 0 i 0

$x < 0$: differenskvoten

$\dfrac{f(x)-f(0)}{x-0} = \dfrac{\sin x - x}{x^2}$ är en udda funktion

$\Rightarrow \exists \lim\limits_{x\to 0_-} = -\lim\limits_{x\to 0_+} = 0$

$\Rightarrow \exists f'(0) = 0$

(b) $x \neq 0$: $f'(x) = \dfrac{x\cos x - \sin x}{x^2}$ (7)

$? \ \exists \lim_{x \to 0} f'(x) = f'(0) = 0$

$$\frac{x\cos x - \sin x}{x^2} = \frac{x\cos x - x + x - \sin x}{x^2}$$

$$= \frac{\cos x - 1}{x} - \frac{\frac{\sin x}{x} - 1}{x} \xrightarrow[x \to 0]{} 0$$

$\frac{\cos x - 1}{x} \xrightarrow[x \to 0]{} 0$, $\frac{\frac{\sin x}{x} - 1}{x} \xrightarrow[x \to 0]{} 0$

end. (a)