

TMA970 Inledande
matematisk analys, FI/TM1
Lösningar 26/10-2017

- ① (a) konvergent (b) divergent
 (c) konvergent (d) divergent (i-i)
 (e) divergent (i $\frac{1}{2}$) (f) divergent

② (a) $\sqrt{x^2+x+1} - \sqrt{x^2-x+1} = \left(\begin{smallmatrix} \text{går mot} \\ \infty - \infty \end{smallmatrix} \right)$

$$= \frac{(\sqrt{x^2+x+1})^2 - (\sqrt{x^2-x+1})^2}{\sqrt{x^2+x+1} + \sqrt{x^2-x+1}} =$$

$$= \frac{2x}{\sqrt{x^2(1+\frac{1}{x}+\frac{1}{x^2})} + \sqrt{x^2(1-\frac{1}{x}+\frac{1}{x^2})}}$$

$$= \frac{2x}{|x| \left(\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} + \sqrt{1-\frac{1}{x}+\frac{1}{x^2}} \right)} = \frac{2x}{-x \left(\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} + \sqrt{1-\frac{1}{x}+\frac{1}{x^2}} \right)}$$

$x \rightarrow -\infty$
 $|x| = -x$

$$\rightarrow \frac{2}{-2} = -1$$

(b) Sätt $\frac{\pi}{3} - x = t$, $x \rightarrow \frac{\pi}{3} \Leftrightarrow t \rightarrow 0$

$$\frac{1 - 2\cos\left(\frac{\pi}{3} - t\right)}{t} = \frac{1 - 2\left(\frac{1}{2}\cos t + \frac{\sqrt{3}}{2}\sin t\right)}{t} =$$

$$\begin{aligned}
 &= \frac{1 - \cos t}{t} - \sqrt{3} \frac{\sin t}{t} = \\
 &= \frac{1 - \cos^2 t}{t(1 + \cos t)} - \sqrt{3} \frac{\sin t}{t} = \frac{\sin^2 t}{t^2} \cdot \frac{t}{1 + \cos t} - \\
 &- \sqrt{3} \frac{\sin t}{t} \xrightarrow{t \rightarrow 0} 0 - \sqrt{3} = -\sqrt{3}
 \end{aligned}$$

3.
2

$$f(x) = \sqrt[3]{1-x}$$

$$D_f: \mathbb{R}$$

Nollställen: $f(x) = 0 \Leftrightarrow x = 1$

tecken: $f(x) > 0 \forall x < 1$; $f(x) < 0 \forall x > 1$
 varken jämn eller udda; ej periodisk
 (∃ symmetri typ "udda" m.a.p. $(1, 0)$)

$D_f = \mathbb{R}$; f kontinuerlig

⇒ inga vertikala asymptoter

Sneda asymptoter?

$$\lim_{x \rightarrow \infty} f(x) = -\infty, \quad \lim_{x \rightarrow -\infty} f(x) = +\infty$$

$$\lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = \lim_{x \rightarrow \pm\infty} \frac{\sqrt[3]{1-x}}{x} = 0$$

$$\text{men } \lim_{x \rightarrow \pm\infty} (f(x) - 0 \cdot x) = \mp\infty$$

⇒ inga sneda asymptoter

$$f'(x) = \frac{1}{3\sqrt[3]{(1-x)^2}} \cdot (-1), \quad \text{ej def i } x = 1$$

$$f' < 0 \quad \forall x \neq 1$$

⇒ f avtagande, inga lok. extrema

($f' \xrightarrow{x \rightarrow 1} -\infty \Rightarrow$ vertikal tangent $x = 1$)

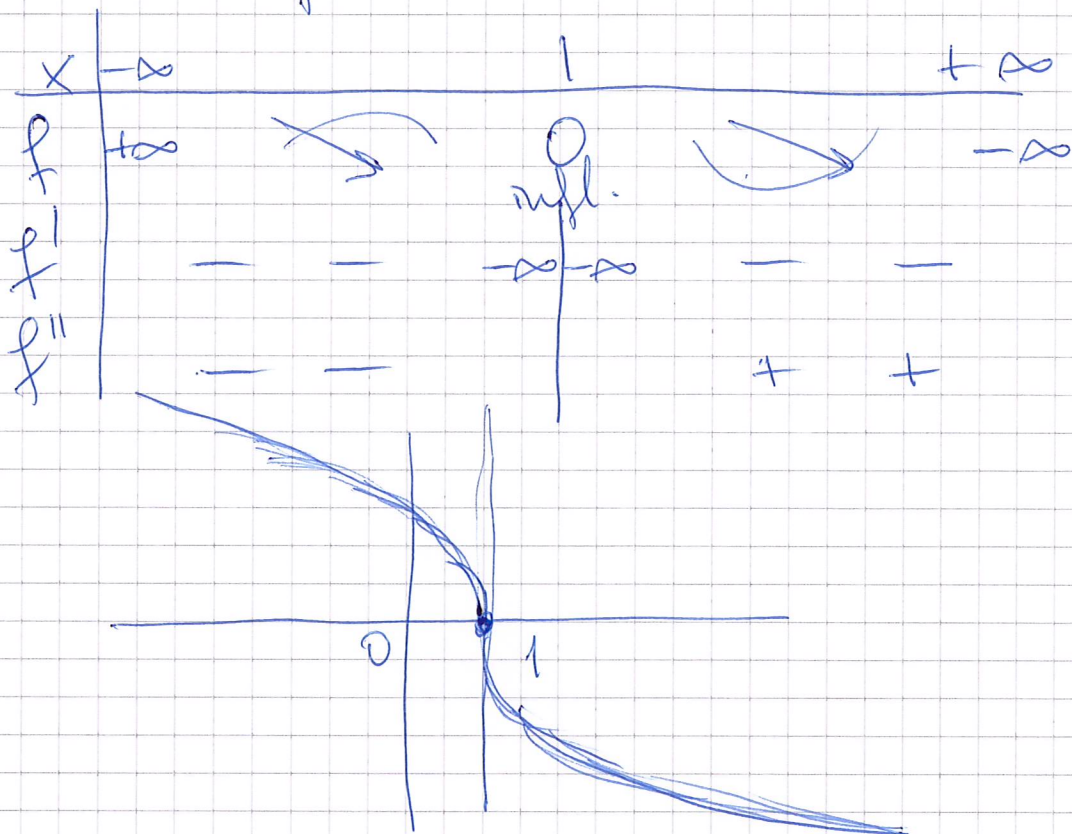
$$f''(x) = \left(-\frac{1}{3} (1-x)^{-2/3} \right)' =$$

$$= \frac{2}{9} (1-x)^{-5/3} \cdot (-1) = -\frac{2}{9} \cdot \frac{1}{\sqrt[3]{(1-x)^5}} =$$

$$= -\frac{2}{9} \cdot \frac{1}{(1-x) \sqrt{(1-x)^2}} \quad (\neq 0)$$

$$\begin{aligned} f'' &> 0 & \forall x > 1 & \quad (\neq f'' : x=1) \\ f'' &< 0 & \forall x < 1 \end{aligned}$$

$\Rightarrow f$ konvex i $(1, \infty)$,
 konkav i $(-\infty, 1)$,
 inflexion i $x=1$



$\textcircled{4} (a) \int \frac{dx}{1+e^x+e^{2x}+e^{3x}} = \left[\begin{array}{l} t=e^x \\ x=\ln t \end{array} ; dx = \frac{1}{t} dt \right]$

$$= \int \frac{dt}{t(1+t+t^2+t^3)} = \int \frac{dt}{t(1+t)(1+t^2)} = \textcircled{x}$$

Partialbråkuppdelning:

4

$$\frac{1}{t(1+t)(1+t^2)} = \frac{A}{t} + \frac{B}{t+1} + \frac{Ct+D}{t^2+1}$$

$$1 = A(t+1)(t^2+1) + Bt(t^2+1) + (Ct+D)t(t+1)$$

$$\begin{cases} t=0 : & 1 = A \\ t=-1 : & 1 = -2B \Rightarrow B = -\frac{1}{2} \\ t^3 : & 0 = A+B+C \Rightarrow C = -\frac{1}{2} \\ t^0 : & 1 = A \quad (\text{konstanten för man} \\ & \text{för } t=0) \end{cases}$$

$$t^2 : \quad 0 = A+C+D \Rightarrow D = -\frac{1}{2}$$

$$\begin{aligned} \Rightarrow \otimes &= \int \frac{dt}{t} - \frac{1}{2} \left(\frac{dt}{t+1} - \frac{1}{2} \left(\frac{2t}{t^2+1} dt - \right. \right. \\ & \left. \left. - \frac{1}{2} \left(\frac{dt}{t^2+1} = \ln|t| - \frac{1}{2} \ln|t+1| - \frac{1}{4} \ln(t^2+1) - \right. \right. \right. \\ & \left. \left. - \frac{1}{2} \arctan t = \right. \right. \\ & = \underline{\ln e^x} - \frac{1}{2} \ln(1+e^x) - \frac{1}{4} \ln(1+e^{2x}) - \\ & = x - \frac{1}{2} \arctan e^x + C \end{aligned}$$

$$\begin{aligned} \text{(b)} \int_0^1 x e^{-2x} dx &= \left[-\frac{1}{2} x e^{-2x} \right]_0^1 + \\ &+ \frac{1}{2} \int_0^1 e^{-2x} \cdot 2x dx = -\frac{1}{2} e^{-2} - \frac{1}{2} \left[x e^{-2x} \right]_0^1 + \\ &+ \frac{1}{2} \int_0^1 e^{-2x} \cdot 1 dx = -\frac{1}{2} e^{-2} - \frac{1}{2} e^{-2} - \frac{1}{4} \left[e^{-2x} \right]_0^1 = \\ &= -e^{-2} - \frac{1}{4} e^{-2} + \frac{1}{4} \end{aligned}$$

5

$$\begin{aligned}
 \textcircled{5} \quad & \left| \sin \sqrt{x+1} - \sin \sqrt{x} \right| = \\
 & = \left| 2 \sin \frac{\sqrt{x+1} - \sqrt{x}}{2} \cos \frac{\sqrt{x+1} + \sqrt{x}}{2} \right| = \\
 & = 2 \underbrace{\left| \sin \frac{x+1-x}{2(\sqrt{x+1} + \sqrt{x})} \right|}_{\xrightarrow{x \rightarrow \infty} 0} \underbrace{\left| \cos \frac{\sqrt{x+1} + \sqrt{x}}{2} \right|}_{\leq 1 \forall x} \xrightarrow{x \rightarrow \infty} 0
 \end{aligned}$$

Alternativ lösning: Eul. medelvärdessatsen
 $\sin \sqrt{x+1} - \sin \sqrt{x} = (\sin \sqrt{x})' \Big|_{x=\xi} (x+1-x) =$

$$\begin{aligned}
 & = \underbrace{\cos \sqrt{\xi_x}}_{\text{begr.}} \cdot \underbrace{\frac{1}{2\sqrt{\xi_x}}}_{\xrightarrow{x \rightarrow \infty} 0} \xrightarrow{x \rightarrow \infty} 0, \\
 & \text{ty } x < \xi_x < x+1 \quad \text{och} \quad \xi_x \xrightarrow{x \rightarrow \infty} \infty
 \end{aligned}$$

$\textcircled{6}$ Tag $F(x) = \int_0^x f(t) dt, x \in (-a, a)$
 Eul. analysens huvudsats $F' = f$

$$\begin{aligned}
 F(-x) & = \int_0^{-x} f(t) dt = \left[\begin{array}{l} t = -s; \\ dt = -ds; \end{array} \right. \left. \begin{array}{l} t=0 \Leftrightarrow s=x \\ t=-x \Leftrightarrow s=0 \end{array} \right] \\
 & = \int_0^x f(-s) (-1) ds = \\
 & = - \int_0^x f(-s) ds = - \int_0^x f(s) ds = -F(x) \\
 & \text{ty } f \stackrel{!}{=} \text{jämn} \Rightarrow F \text{ udda}
 \end{aligned}$$

Alla primitiva: $\phi(x) = F(x) + C$
 $\phi(x) = F(-x) + C = -F(x) - C + 2C; \text{ udda} \Leftrightarrow C=0$