TMA970 Inledande matematisk analys, F1/TM1

Lösningar 26/10-2017

1. (a) konvergent (b) divergent
(c) konvergent (d) divergent (i -1)
(e) divergent (i $\frac{1}{2}$) (f) divergent

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2. (a) $\sqrt{x^2+x+1} - \sqrt{x^2-x+1} = (\text{"}\infty - \infty\text{"})$ går mot

$$= \frac{(\cancel{x^2}+x+\cancel{1}) - (\cancel{x^2}-x+\cancel{1})}{\sqrt{x^2+x+1}+\sqrt{x^2-x+1}} =$$

$$= \frac{2x}{\sqrt{x^2\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}+\sqrt{x^2\left(1-\frac{1}{x}+\frac{1}{x^2}\right)}} =$$

$$= \frac{2x}{|x|\left(\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}\right)} \underset{x\to-\infty}{\overset{|x|=-x}{=}} \frac{2\cancel{x}}{-\cancel{x}\left(\sqrt{1+\underbrace{\frac{1}{x}+\frac{1}{x^2}}_{\to 0}}+\sqrt{1\underbrace{-\frac{1}{x}+\frac{1}{x^2}}_{\to 0}}\right)}$$

$$\longrightarrow \frac{2}{-2} = -1$$

(b) Sätt $\frac{\pi}{3} - x = t$, $x \to \frac{\pi}{3} \iff t \to 0$

$$\frac{1-2\cos\left(\frac{\pi}{3}-t\right)}{t} = \frac{1-\cancel{2}\left(\frac{1}{\cancel{2}}\cos t + \frac{\sqrt{3}}{\cancel{2}}\sin t\right)}{t} =$$

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$$= \frac{1-\cos t}{t} - \sqrt{3}\,\frac{\sin t}{t} =$$

$$= \frac{1-\cos^2 t}{t(1+\cos t)} - \sqrt{3}\,\frac{\sin t}{t} = \underbrace{\frac{\sin^2 t}{t^2}}_{\xrightarrow[t\to 0]{} 1} \cdot \frac{\overbrace{t}^{\to 0}}{\underbrace{1+\cos t}_{\xrightarrow[t\to 0]{} 2}} -$$

$$- \sqrt{3}\,\underbrace{\frac{\sin t}{t}}_{\xrightarrow[t\to 0]{} 1} \xrightarrow[t\to 0]{} 0 - \sqrt{3} = -\sqrt{3}$$

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3. $f(x) = \sqrt[3]{1-x}$ $\quad D_f : \mathbb{R}$

Nollställen: $f(x) = 0 \iff x = 1$

tecken: $f(x) > 0 \ \forall x < 1$; $f(x) < 0 \ \forall x > 1$

varken jämn eller udda; ej periodisk

($\exists$ symmetri typ "udda" m.a.p. $(1,0)$)

$D_f = \mathbb{R}$; f kontinuerlig

$\Rightarrow$ inga vertikala asymptoter

Sneda asymptoter?

$\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = +\infty$

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{\sqrt[3]{1-x}}{x} = 0$$

men $\lim_{x\to\pm\infty} (f(x) - 0\cdot x) = \mp\infty$

$\Rightarrow$ inga sneda asymptoter

$$f'(x) = \frac{1}{3\sqrt[3]{(1-x)^2}} \cdot (-1), \text{ ej def i } x = 1$$

$f' < 0 \quad \forall x \neq 1$

$\Rightarrow f$ avtagande, inga lok. extrema

($f' \xrightarrow[x\to 1]{} -\infty \Rightarrow$ vertikal tangent $x=1$)

$$f''(x) = \left(-\frac{1}{3}(1-x)^{-2/3}\right)' =$$

$$= \frac{2}{9}(1-x)^{-5/3} \cdot (-1) = -\frac{2}{9} \cdot \frac{1}{\sqrt[3]{(1-x)^5}} =$$

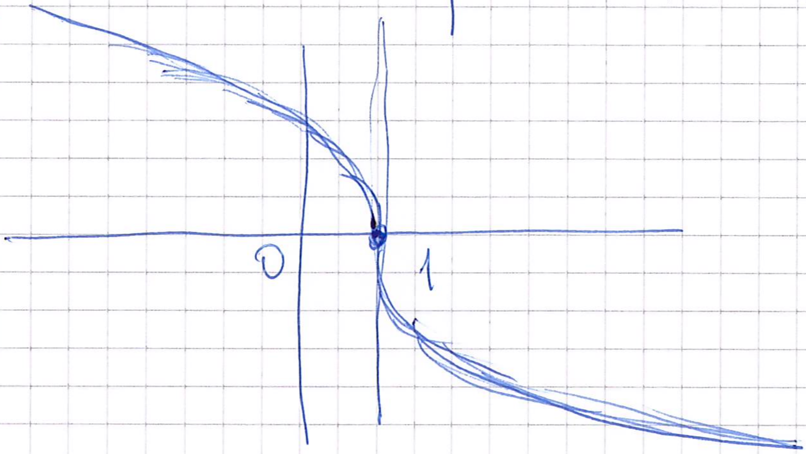
$$= -\frac{2}{9} \cdot \frac{1}{(1-x)\sqrt[3]{(1-x)^2}} \quad (\neq 0)$$

$f'' > 0 \quad \forall x > 1$

$f'' < 0 \quad \forall x < 1 \quad (\nexists f'' \text{ i } x = 1)$

$\Rightarrow f$ konvex i $(1, \infty)$,
konkav i $(-\infty, 1)$,
inflexion i $x = 1$

x	$-\infty$		1		$+\infty$
f	$+\infty$	$\searrow$	0 infl.	$\searrow$	$-\infty$
f'		$-\ \ -$	$-\infty \mid -\infty$	$-\ \ -$	
f''		$-\ \ -$		$+\ \ +$	



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(4) (a) $\displaystyle\int \frac{dx}{1+e^x+e^{2x}+e^{3x}} = \left[\begin{array}{l} t = e^x \\ x = \ln t \end{array} ; \ dx = \frac{1}{t}dt\right]$

$$= \int \frac{dt}{t(1+t+t^2+t^3)} = \int \frac{dt}{t(1+t)(1+t^2)} = \circledast$$

Partialbråksuppdelning:

$$\frac{1}{t(1+t)(1+t^2)} = \frac{A}{t} + \frac{B}{t+1} + \frac{Ct+D}{t^2+1}$$

$$1 = A(t+1)(t^2+1) + Bt(t^2+1) + (Ct+D)t(t+1)$$

$t=0$: $1 = A$

$t=-1$: $1 = -2B \Rightarrow B = -\frac{1}{2}$

t^3 : $0 = A+B+C \Rightarrow C = -\frac{1}{2}$

t^0 : $1 = A$ (konstanten får man för $t=0$)

t^2 : $0 = A+C+D \Rightarrow D = -\frac{1}{2}$

$$\Rightarrow (*) = \int \frac{dt}{t} - \frac{1}{2}\int \frac{dt}{t+1} - \frac{1}{2\cdot 2}\int \frac{2t}{t^2+1}\,dt - \frac{1}{2}\int \frac{dt}{t^2+1} = \ln|t| - \frac{1}{2}\ln|t+1| - \frac{1}{4}\ln(t^2+1) - \frac{1}{2}\arctan t =$$

$$= \underbrace{\ln e^x}_{=x} - \frac{1}{2}\ln(1+e^x) - \frac{1}{4}\ln(1+e^{2x}) - \frac{1}{2}\arctan e^x + C$$

(b) $$\int_0^1 x^2 e^{-2x}\,dx = \left[-\frac{1}{2}x^2 e^{-2x}\right]_0^1 + \frac{1}{2}\int_0^1 e^{-2x}\cdot 2x\,dx = -\frac{1}{2}e^{-2} - \frac{1}{2}\left[xe^{-2x}\right]_0^1 + \frac{1}{2}\int_0^1 e^{-2x}\cdot 1\,dx = -\frac{1}{2}e^{-2} - \frac{1}{2}e^{-2} - \frac{1}{4}\left[e^{-2x}\right]_0^1 =$$

$$= -e^{-2} - \frac{1}{4}e^{-2} + \frac{1}{4}$$

5. $\left|\sin\sqrt{x+1} - \sin\sqrt{x}\right| =$

$$= \left|2\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}\cos\frac{\sqrt{x+1}+\sqrt{x}}{2}\right| =$$

$$= 2\underbrace{\left|\sin\frac{\cancel{x}+1-\cancel{x}}{2(\sqrt{x+1}+\sqrt{x})}\right|}_{\xrightarrow[x\to\infty]{} 0}\underbrace{\left|\cos\frac{\sqrt{x+1}+\sqrt{x}}{2}\right|}_{\leq 1 \ \forall x} \xrightarrow[x\to\infty]{} 0$$

Alternativ lösning: Enl. medelvärdessatsen

$$\sin\sqrt{x+1} - \sin\sqrt{x} = (\sin\sqrt{x})'\Big|_{x=\xi}(\cancel{x}+1-\cancel{x}) =$$

$$= \underset{\text{begr.}}{\cos\sqrt{\xi_x}} \cdot \frac{1}{2\sqrt{\xi_x}} \xrightarrow[x\to\infty]{} 0, \quad \left(\frac{1}{2\sqrt{\xi_x}} \to 0\right)$$

ty $x < \xi_x < x+1$ o $\xi_x \xrightarrow[x\to\infty]{} \infty$

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6. Tag $F(x) = \int_0^x f(t)\,dt, \quad x \in (-a,a)$

Enl. analysens huvudsats $F' = f$

$$F(-x) = \int_0^{-x} f(t)\,dt = \begin{bmatrix} t = -s & ; & t = -x \Leftrightarrow s = x \\ dt = -ds & ; & t=0 \Leftrightarrow s=0 \end{bmatrix}$$

$$= \int_0^x f(-s)(-1)\,ds =$$

$$= -\int_0^x f(-s)\,ds \underset{\uparrow}{=} -\int_0^x f(s)\,ds = -F(x)$$

ty f jämn $\Rightarrow F$ udda

Alla primitiva: $\Phi(x) = F(x) + C$

$\Phi(-x) = F(-x) + C = -F(x) - C + 2C$; udda $\Leftrightarrow C = 0$