TMA Inledande matematisk analys F1 / TM1

Lösningar 31/8 -18

1. (a) divergent; (b) divergent; (c) konvergent; (d) konvergent; (e) divergent; (f) divergent.

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2. (a)
$$\frac{e^{x^2-x}-e^{-x}}{e^{x^2+x}+e^{x}} = \frac{e^{-x}\left(e^{x^2}-1\right)}{e^{x}\left(e^{x^2}+1\right)} =$$

$$= \frac{1}{e^{2x}} \cdot \frac{1-\frac{1}{e^{x^2}}}{1+\frac{1}{e^{x^2}}} \xrightarrow[x\to\infty]{} 0 \cdot \frac{1-0}{1+0} = 0$$

(b)
$$\frac{\cos x + 1}{\sin x \cdot \tan x} = \left[\begin{array}{l} x = t+\pi \\ x \to \pi \iff t \to 0 \end{array}\right]$$

$$= \frac{\cos(t+\pi)+1}{\frac{\sin^2(t+\pi)}{\cos(t+\pi)}} = -\cos t \cdot \frac{1-\cos t}{(-\sin t)^2} =$$

$$= -\cos t \cdot \frac{2\cancel{\sin^2\frac{t}{2}}}{4\cancel{\sin^2\frac{t}{2}}\cos^2\frac{t}{2}} \longrightarrow -1 \cdot \frac{2}{4\cdot 1} =$$

$$= -\frac{1}{2}$$

3. $D_f = \mathbb{R}$ $\left(f(x) = \frac{x-1}{\sqrt{x^2+4}}\right)$ ②

Nollställen: endast $x=1$

Tecken: $f(x) < 0 \quad \forall x < 1$,

$> 0 \quad \forall x > 1$

Inga symmetrier

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x\left(1-\overbrace{\frac{1}{x}}^{\to 0}\right)}{|x|\sqrt{1+\underbrace{\frac{4}{x^2}}_{\to 0}}} = \pm 1$$

$\Rightarrow$ f har horisontella asymptoter i $\pm\infty$, som är $y = \pm 1$

Inga vertikala asymptoter

$$f'(x) = \frac{1\cdot\sqrt{x^2+4} - \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+4}}\cdot(x-1)}{x^2+4} =$$

$$= \frac{\cancel{x^2}+4-\cancel{x^2}+x}{(x^2+4)^{3/2}} = \frac{x+4}{(x^2+4)^{3/2}}$$

x	$-\infty$	-4	$+\infty$
f'	$-$	0	$+$

$\Rightarrow$ f har lok. min i $x=-4$

$$f''(x) = \frac{1\cdot(x^2+4)^{3/2} - \frac{3}{\cancel{2}}(x^2+4)^{1/2}\cdot\cancel{2}x\cdot(x+4)}{(x^2+4)^3} =$$

$$= \frac{x^2+4-3x(x+4)}{(x^2+4)^{5/2}} = \frac{-2x^2-12x+4}{(x^2+4)^{5/2}}$$

$f'' = 0$: $x^2 + 6x - 2 = 0$

$$x_{1,2} = \frac{-3 \pm \sqrt{11}}{1} = -3 \pm \sqrt{11}$$

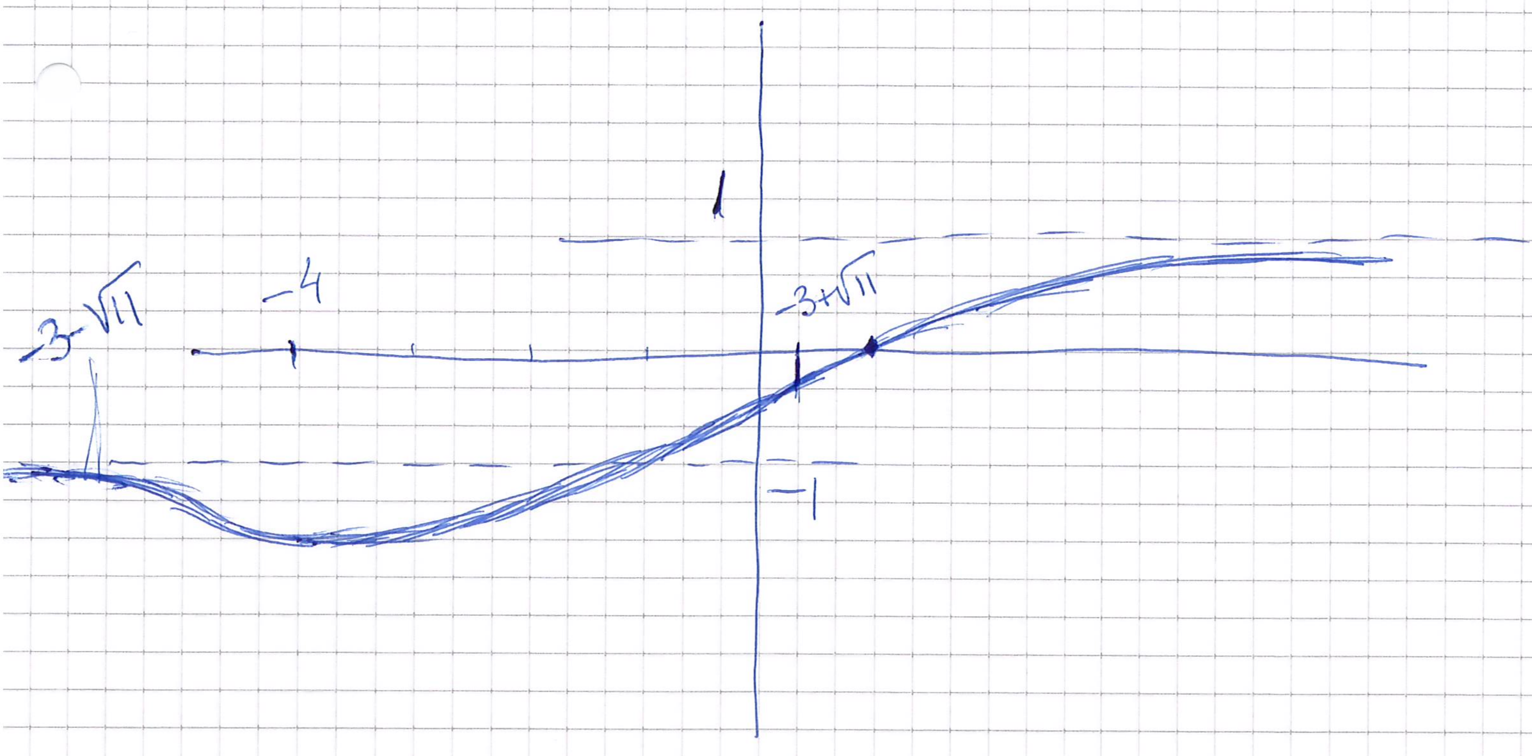
nämnaren i f'' är > 0 ;

täljaren :

x	$-\infty$	$-3-\sqrt{11}$ (mellan -7 och -6)		$-3+\sqrt{11}$ (mellan 0 och 1)	$+\infty$
f''	$-$	0	$+$	0	$-$

$\Rightarrow$ f har två inflexionspkter, eftersom f'' byter tecken i sina nollställen

x	$-\infty$		$-3-\sqrt{11}$	-4	$-3+\sqrt{11}$	1		$+\infty$
f	-1		infl.	lok.min	infl.	0		$+1$
f'		$-$	$-$	0	$+$		$+$	
f''		$-$	0	$+$	0		$-$	



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4. (a) $\int x \ln\left(1+\frac{1}{x}\right) dx \overset{\text{p.i.}}{=}$

$x > 0$

$$= \frac{x^2}{2}\ln\left(1+\frac{1}{x}\right) - \int \frac{x^2}{2}\cdot\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)dx =$$

$$= \frac{1}{2}x^2\ln\left(1+\frac{1}{x}\right) + \frac{1}{2}\int\frac{x\,dx}{x+1} =$$

$$= \frac{1}{2}x^2\ln\left(1+\frac{1}{x}\right) + \frac{1}{2}\int\frac{x+1-1}{x+1}dx =$$

$$= \frac{1}{2}x^2\ln\left(1+\frac{1}{x}\right) + \frac{1}{2}x - \frac{1}{2}\ln(x+1) \quad (+C)$$

(b) Sätt $t^3 = \frac{2-x}{2+x}$

$2t^3 + xt^3 = 2 - x, \quad x = \frac{2-2t^3}{1+t^3}$

$$dx = \frac{-6t^2(1+t^3) - 3t^2(2-2t^3)}{(1+t^3)^2}dt =$$

$$= \frac{-6t^2 - \cancel{6t^5} - 6t^2 + \cancel{6t^5}}{(1+t^3)^2}dt = \frac{-12t^2}{(1+t^3)^2}dt$$

$$2 - x = 2 - \frac{2-2t^3}{1+t^3} = \frac{\cancel{2} + 2t^3 - \cancel{2} + 2t^3}{1+t^3}$$

$x=0$: $t=1$, $x=1$: $t = \frac{1}{\sqrt[3]{3}}$

$$\text{integralen} = \int_1^{1/\sqrt[3]{3}} \frac{2}{\left(\frac{4t^3}{\cancel{1+t^3}}\right)^2}\cdot t\cdot\frac{-12t^2}{\cancel{(1+t^3)^2}}dt =$$

$$= \int_{\sqrt[3]{\frac{1}{3}}}^{1}\frac{3}{2t^3}dt = -\frac{3}{4}\left[\frac{1}{t^2}\right]_{\frac{1}{\sqrt[3]{3}}}^{1} = +\frac{3}{4}\left(-1+3^{2/3}\right)$$

(5.) $F'(x) = \frac{\sin x}{x} = 0$ $\triangle 3$

för $x = k\pi,\ k \in \mathbb{N}$

$x > 0$; sin byter tecken i $k\pi$

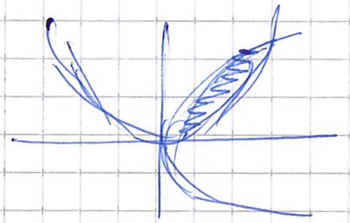
$F' > 0$ i $(0, \pi)$, < 0 i $(\pi, 2\pi)$

induktivt:

F har lok. max i $x = 2n\pi + \pi$

lok. min i $x = 2n\pi$

(6.)



$p > 0$

Vi tar fram skärningspunkterna:

$$\begin{cases} y^2 = 2px \\ x^2 = 2py \end{cases} \qquad x = \frac{1}{2p} y^2$$

$$\frac{1}{(2p)^2} y^4 = 2py$$

$$y^4 = (2p)^3 y \qquad y\left(y^3 - (2p)^3\right) = 0$$

$y_1 = 0$ ger $x = 0$

$y_2 = 2p$ ger $x = 2p$

$$\sqrt{2px} - \frac{x^2}{2p} = \sqrt{2px}\left(1 - \frac{x^{3/2}}{(2p)^{3/2}}\right) \geq 0 \qquad \underline{0 \leq x \leq 2p}$$

$\Rightarrow$ parabeln $y^2 = 2px$ ligger överst i det intervallet

$$\Rightarrow \text{arean} = \int_0^{2p} \left(\sqrt{2px} - \frac{x^2}{2p} \right) dx =$$

$$= \left[\sqrt{2p} \cdot \frac{x^{3/2}}{3/2} - \frac{1}{2p} \cdot \frac{1}{3} x^3 \right]_0^{2p} =$$

$$= \frac{2}{3} \sqrt{2p} \cdot \left(\sqrt{2p}\right)^{3/2} - \frac{1}{3} \cdot \frac{1}{\cancel{2p}} \cdot (2p)^{\cancel{3}\,2} - 0 + 0$$

$$= \frac{2}{3} (2p)^2 - \frac{1}{3} \cdot (2p)^2 = \frac{4}{3} p^2$$