TMA970 Inledande matematisk analys F1/TM1

Lösningar 31/10-2018

1. (a) divergent; (b) divergent; (c) konvergent; (d) konvergent; (e) konvergent; (f) konvergent.

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2. (a) $\sqrt{x+\sqrt{x}} - \sqrt{x} =$

$$= \frac{\left(\sqrt{x+\sqrt{x}} - \sqrt{x}\right)\left(\sqrt{x+\sqrt{x}} + \sqrt{x}\right)}{\sqrt{x+\sqrt{x}} + \sqrt{x}} =$$

$$= \frac{\cancel{x} + \sqrt{x} - \cancel{x}}{\sqrt{x}\left(\sqrt{1+\frac{1}{\sqrt{x}}} + 1\right)} = \frac{1}{\sqrt{1+\frac{1}{\sqrt{x}}} + 1} \xrightarrow[x\to\infty]{} \frac{1}{2}$$

(b) $$\frac{\sqrt{1+\tan x} - \sqrt{1+\sin x}}{x^3} =$$

$$= \frac{\cancel{1} + \tan x - \cancel{1} - \sin x}{x^3\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)} =$$

$$= \frac{\sin x}{x} \cdot \frac{\frac{1}{\cos x} - 1}{x^2} \cdot \frac{1}{\sqrt{1+\tan x} + \sqrt{1+\sin x}} =$$

$$= \frac{\sin x}{x} \cdot \frac{1}{\cos x} \cdot \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} \cdot \frac{1}{\sqrt{1+\tan x} + \sqrt{1+\sin x}}$$

$\left((1-\cos x)(1+\cos x) = 1 - \cos^2 x = \sin^2 x \right)$

$$= \frac{\sin x}{x} \cdot \frac{1}{\cos x} \cdot \frac{\sin^2 x}{x^2} \cdot \frac{1}{1+\cos x} \cdot \frac{1}{\sqrt{1+\tan x}+\sqrt{1+\sin x}}$$ (2)

$$\longrightarrow 1 \cdot \frac{1}{1} \cdot 1^2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

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(3) $f(x) = \frac{\ln x}{\sqrt{x}}$ $\qquad D_f : x > 0$

Nollställen: $f(x) = 0 \Leftrightarrow \ln x = 0$
$\Leftrightarrow x = 1$

$$\left.\begin{array}{ll} f(x) < 0 & \text{i } (0,1) \\ f(x) > 0 & \text{i } (1,\infty) \end{array}\right\} \text{tecken}$$

$$\lim_{x \to 0_+} \frac{\ln x}{\sqrt{x}} = \text{"}\frac{-\infty}{+0}\text{"} = -\infty$$

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} = 0 \quad \text{(standardgränsvärde)}$$

inga vertikala asymptoter
sned (horisontell) asymptot
i $+\infty$: $y = 0$

$$f'(x) = \frac{\frac{1}{x}\sqrt{x} - \frac{1}{2\sqrt{x}}\ln x}{x} =$$

$$= \frac{2 - \ln x}{2x^{3/2}} = 0 \text{ för } x = e^2$$

x	0		e^2	
f'		$+$	0	$-$

$\Rightarrow$ f växande i $(0, e^2)$, avtagande i (e^2, ∞), lokalmax i $x = e^2$

$\triangle 3$

$$f''(x) = \frac{-\frac{1}{\cancel{x}}\cancel{2}x^{\cancel{3/2}\,1/2} - \cancel{2}\cdot\frac{3}{2}\cdot\sqrt{x}\,(2-\ln x)}{2\cancel{4}x^3}$$

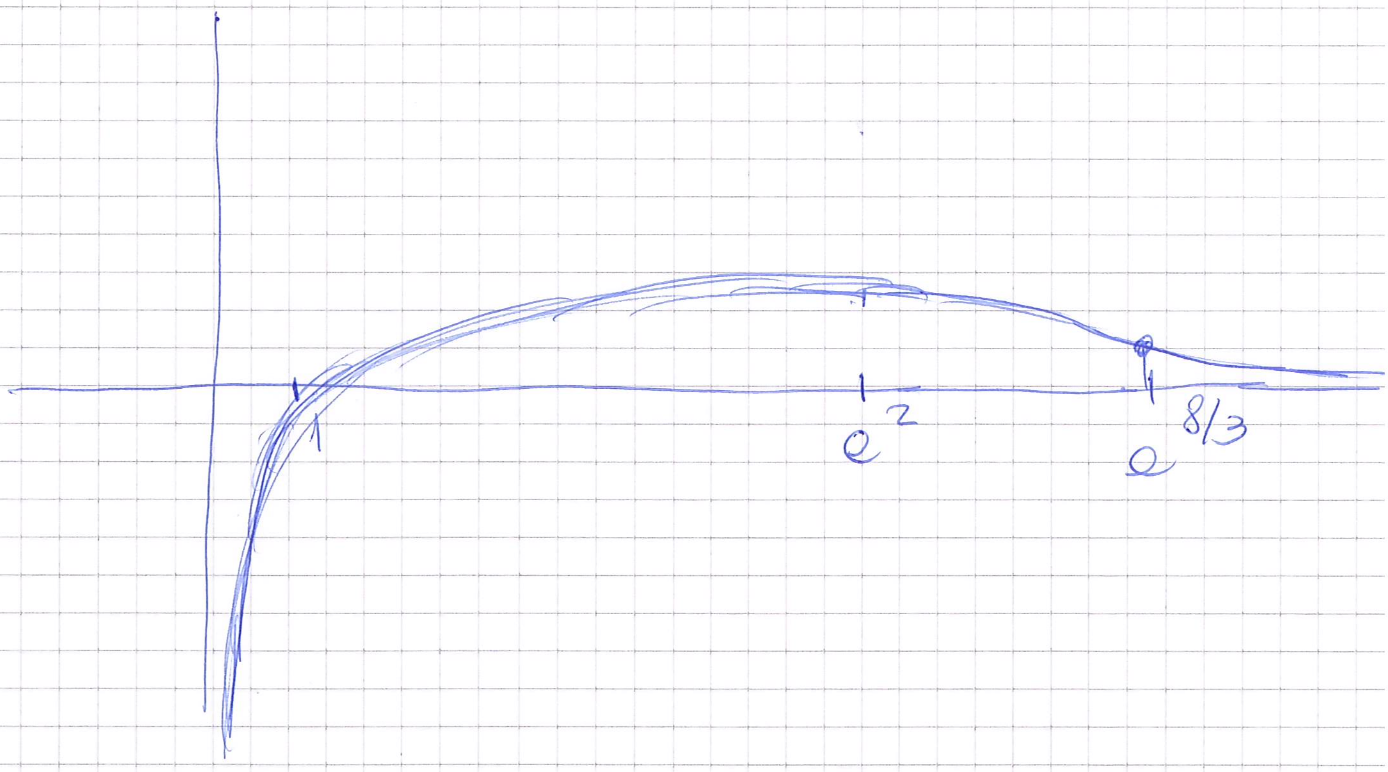
$$= \frac{-1-3+\frac{3}{2}\ln x}{2x^{5/2}} = \frac{3\ln x - 8}{4x^{5/2}}$$

$$f'' = 0 \iff x = e^{8/3}$$

x	0		$e^{8/3}$	
f''		$-$	0	$+$

x	0	1	e^2	$e^{8/3}$	$+\infty$
f	$-\infty$	0	loc max $2/e$		0
f'	$+$	$+$	0	$-$	$-$
f''	$-$	$-$		0	$+$

inflexion



4. (a) $\displaystyle\int \frac{\cos x}{2+\cos 2x}\,dx =$ $\boxed{4}$

$$= \int \frac{(\sin x)'}{\sqrt{3-2\sin^2 x}}\,dx = \left[\begin{matrix}\sin x = t\\ (\sin x)'\,dx = dt\end{matrix}\right] =$$

$$= \int \frac{dt}{\sqrt{3-2t^2}} = \frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{1-\left(\sqrt{\frac{2}{3}}\,t\right)^2}} =$$

$$= \frac{1}{\cancel{\sqrt{3}}}\cdot\sqrt{\frac{\cancel{3}}{2}}\arcsin\sqrt{\frac{2}{3}}\,t \quad (+C) =$$

$$= \frac{1}{\sqrt{2}}\arcsin\left(\sqrt{\frac{2}{3}}\sin x\right) \quad (+C)$$

(b) $\displaystyle\int_0^{1-\varepsilon} \arccos x\,dx = \Big[x\arccos x\Big]_0^{1-\varepsilon} -$

$$- \int_0^{1-\varepsilon} \frac{x}{\sqrt{1-x^2}}(-1)\,dx =$$

generaliserad! $\int_0^1$

$$= (1-\varepsilon)\arccos(1-\varepsilon) - 0 +$$

$$+ \frac{1}{2}\int_0^{1-\varepsilon} \frac{(x^2)'}{\sqrt{1-x^2}}\,dx = (1-\varepsilon)\arccos(1-\varepsilon) -$$

$$- \frac{1}{2}\int_0^{1-\varepsilon} \frac{(1-x^2)'}{\sqrt{1-x^2}}\,dx =$$

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$$= (1-\varepsilon)\arccos(1-\varepsilon) - \frac{1}{\cancel{2}}\cdot\cancel{2}\left[\sqrt{1-x^2}\right]_0^{1-\varepsilon}$$

$$\xrightarrow[\varepsilon\to 0]{} \arccos 1 - \sqrt{1-1^2} + \sqrt{1-0^2} =$$

$$= 0 - 0 + 1$$

OBS! Båda deluppgifterna kan göras på andra sätt, t ex.
(a) med universalsubstitutionen, och
(b) med substitutionen $x = \cos t$.

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5. (i) $|\sin x - \sin y| = \left|2\sin\frac{x-y}{2}\cos\frac{x+y}{2}\right|$

$$\leq 2\left|\sin\frac{x-y}{2}\right|$$

För $\frac{|x-y|}{2} \leq \frac{\pi}{2}$ har vi

$$2\left|\sin\frac{x-y}{2}\right| = 2\sin\frac{|x-y|}{2} \leq \cancel{2}\,\frac{|x-y|}{\cancel{2}}$$

För $\frac{|x-y|}{2} \geq \frac{\pi}{2} (>1)$ har vi

$$2\left|\sin\frac{x-y}{2}\right| \leq 2 \leq \cancel{2}\cdot\frac{|x-y|}{\cancel{2}}$$

(ii) Enligt Lagranges medelvärdessats
$\exists\ \xi$ mellan x och y s.a. ($f = \sin$)

$(x \neq y)$ $|\sin x - \sin y| = \underbrace{|f'(\xi)|}_{=|\cos\xi|}\,|x-y| \leq |x-y|$

För $x = y$: $0 = 0$, gäller fortfarande.

(6)

6. Om $f(x_1) = \ldots = f(x_n)$,
så kan ξ väljas som vilken som helst av punkterna x_k, $k=1\ldots n$.

Om $\exists\, k, l : f(x_k) \neq f(x_l)$:
utan inskränkning kan vi anta
att $f(x_1) = \min(f(x_1), \ldots, f(x_n))$,
och att $f(x_n) = \max(f(x_1), \ldots, f(x_n))$

$$\frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n} \in (f(x_1), f(x_n))$$

$\Rightarrow$ enligt satsen om mellanliggande värden

$\exists\, \xi$ mellan x_1 och x_n s.a.

$$f(\xi) = \frac{f(x_1) + \ldots + f(x_n)}{n}$$

$x_1, x_2 \in (a, b)$, intervall

$\Rightarrow \xi \in (a, b)$