Låt $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m, \quad \mathbf{y} = \mathbf{f}(\mathbf{x})$.

Funktionalmatrisen:

$$\mathbf{f}' = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \dots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

$$\mathbf{f}(\mathbf{a}+\mathbf{h}) \approx \mathbf{f}(\mathbf{a}) + \mathbf{f}'(\mathbf{a})\mathbf{h} \quad \text{för små } |\mathbf{h}|$$

Kedjeregeln: Antag $\mathbf{y} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} = \mathbf{g}(\mathbf{t})$, då $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{t} \in \mathbb{R}^q$, $(\mathbf{f} \circ \mathbf{g})(\mathbf{t}) = \mathbf{f}(\mathbf{g}(\mathbf{t}))$. Då är

$$(\mathbf{f} \circ \mathbf{g})' = \mathbf{f}'\mathbf{g}'$$

Bevis:

$$[(\mathbf{f} \circ \mathbf{g})']_{jk} = \frac{\partial y_j}{\partial t_k} = \frac{\partial y_j}{\partial x_1}\frac{\partial x_1}{\partial t_k} + \dots + \frac{\partial y_j}{\partial x_n}\frac{\partial x_n}{\partial t_k} = [\mathbf{f}'\mathbf{g}']_{jk}$$

Låt $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n, \quad \mathbf{y} = \mathbf{f}(\mathbf{x})$.

Funktionaldeterminanten (eller Jacobi-determinanten):

$$\det \mathbf{f}'(\mathbf{x}) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \ldots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ldots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \ldots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

Betecknas

$$\frac{d(\mathbf{f})}{d(\mathbf{x})} = \frac{d(f_1, \ldots, f_n)}{d(x_1, \ldots, x_n)} = \frac{d(y_1, \ldots, y_n)}{d(x_1, \ldots, x_n)}$$

Exempel: Polära koordinater i planet:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

$$\frac{d(x,y)}{d(r,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix}$$
$$= r\cos^2\varphi + r\sin^2\varphi = r$$

Exempel: Rymdpolära koordinater:

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$$\frac{d(x,y,z)}{d(r,\theta,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cdots = r^2\sin\theta$$

Av $(\mathbf{f} \circ \mathbf{g})' = \mathbf{f}'\mathbf{g}'$ följer $\det(\mathbf{f} \circ \mathbf{g})' = \det \mathbf{f}' \cdot \det \mathbf{g}'$ eller

$$\frac{d(y_1, \ldots, y_n)}{d(t_1, \ldots, t_n)} = \frac{d(y_1, \ldots, y_n)}{d(x_1, \ldots, x_n)} \cdot \frac{d(x_1, \ldots, x_n)}{d(t_1, \ldots, t_n)}$$

Om $\mathbf{y} = \mathbf{f}(\mathbf{x})$ har en invers $\mathbf{x} = \mathbf{g}(\mathbf{y})$, så är $\mathbf{y} = \mathbf{f}(\mathbf{g}(\mathbf{y}))$,

$$1 = \det E = \frac{d(\mathbf{y})}{d(\mathbf{y})} = \frac{d(\mathbf{y})}{d(\mathbf{x})} \cdot \frac{d(\mathbf{x})}{d(\mathbf{y})}$$

$$\frac{d(\mathbf{x})}{d(\mathbf{y})} = \frac{1}{\dfrac{d(\mathbf{y})}{d(\mathbf{x})}}$$