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Hankel matrices

Hankel matrices are similar to Toeplitz matrices, but 'striped' along the anti-diagonals (SW-NE):

> A matrix is Hankel if all elements on any anti-diagonal are the same.

Example 1: Here is a 3×2 Hankel matrix:

$$\begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 3 & 4 \end{bmatrix}$$

Note 1: Toeplitz and Hankel matrices are related by a mirror imaging — horizontal or vertical flip in image processing terms.

Note 2: Let T be Toeplitz, H Hankel, and I_A the flipped identity matrix (unity on the main anti-diagonal). Then,

$$T\,I_A = H \text{ and } H\,I_A = T.$$

Example 2: Compute

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Hankel matrices appear in signal processing problems. An example is linear and circular correlations.

Example 3: Given the correlation sum

$$r_n = \sum_{k=0}^{K-1} y_k\, x_{n+k},$$

compute the correlation vector $\mathbf{r}$:

$$\mathbf{r} = \begin{bmatrix} r_L \\ \vdots \\ \vdots \\ r_M \end{bmatrix} = \begin{bmatrix} x_L & x_{L+1} & x_{L+2} & \cdots & x_{L+K-1} \\ x_{L+1} & x_{L+2} & & & \\ x_{L+2} & & & & \\ \vdots & & & & \\ x_M & & & & \end{bmatrix} \begin{bmatrix} y_0 \\ \vdots \\ \vdots \\ y_{K-1} \end{bmatrix}$$

The matrix involved is Hankel.

Example 4: Circular correlations relate to linear correlations as circular convolutions relate to linear convolutions. Given x_n, y_n on the interval $n \in [0, N-1]$, compute the circular correlation by letting one data set be periodically repeated:

$$r_n = \sum_{k=0}^{N-1} y_k \, x_{mod_N(n+k)}.$$

This is accomplished for $n = 0, \ldots, N-1$ by

$$\mathbf{r} = \begin{bmatrix} r_0 \\ \vdots \\ \vdots \\ r_{N-1} \end{bmatrix} = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{N-1} \\ x_1 & x_2 & & & x_0 \\ x_2 & & & & \\ \vdots & & & & \\ x_{N-1} & & & & \end{bmatrix} \begin{bmatrix} y_0 \\ \vdots \\ \vdots \\ y_{N-1} \end{bmatrix}$$

The matrix above is circulant Hankel.

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Circulant Hankel matrices can be diagonalized in an interesting way by the Fourier matrix. You can use the same methodology as we did when demonstrating the corresponding property for Toeplitz matrices, but we will take another route. Let T and H denote circulant Toeplitz and Hankel matrices, respectively. If we need more than one, we will index the notation. Start by proving the following lemma:

- $T_1 \cdot T_2 = T$ or in words, the product of two circulant Toeplitz matrices is circulant Toeplitz.

Multiply from the right by I_A:

$$T_1 \, T_2 \, I_A = T \, I_A$$

$\iff$

$T_1 \, H_2 = H$ or in words, the product of a circulant Toeplitz matrix and a circulant Hankel matrix is circulant Hankel.

Transpose the above to show

- $H_2 \cdot T_1 = H$ or in words, the product of a circulant Hankel matrix and a circulant Toeplitz matrix is circulant Hankel.

Right multiply by I_A to show

- $H_2 \cdot H_1 = T$ or in words, the product of two circulant Hankel matrices is circulant Hankel.

Now, make use of the structure of the square of the Fourier matrix:

$$F^2 = \begin{bmatrix} 1 & 0 & \cdots\cdots & 0 \\ 0 & \text{\Large 0} & & 1 \\ \vdots & & \iddots & \\ 0 & 1 & \text{\Large 0} & \end{bmatrix},$$

which is circulant Hankel and a permutation matrix. Note that $F^4 = I$. We find

$$\begin{aligned} D =& FTF^H = FHI_AF^H = FHI_AF^4F^H = \\ =& FH\left(I_AF^2\right)\left(F^2F^H\right) = FH \underset{\substack{\uparrow \\ \text{circulant Toeplitz}}}{\begin{bmatrix} 0\;1\;0 & \cdots & 0 \\ \ddots & & \\ & \ddots & 1 \\ 1\;0 & \cdots & 0 \end{bmatrix}} F = \\ =& FH_1F \end{aligned}$$

The conclusion is

$$\boxed{FAF \text{ is diagonal iff } A \text{ is circulant Hankel.}}$$

As before, F could be replaced by F^H.

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The eigenvalues of circulant Hankel matrices are easy to find:

$$\begin{aligned} |\det(H - \lambda I)| &= \left|\det F^2\right| \cdot |\det(H - \lambda I)| = \\ &= \left|\det\left(FHF - \lambda F^2\right)\right| = \left|\det\left(D - \lambda F^2\right)\right| = \\ &= \left|\det \begin{bmatrix} d_0 & -\lambda & 0 & \cdots & \cdots & 0 \\ 0 & & d_1 & & & -\lambda \\ \vdots & & & \ddots & \iddots & \\ \vdots & & & \iddots & \ddots & \\ 0 & & -\lambda & & & d_{N-1} \end{bmatrix}\right| \end{aligned}$$

Thus, the eigenvalues of H are (notice that after proper permutations, the matrix above is block-diagonal with blocks of size 1×1 or 2×2):

$$\lambda_k = \begin{cases} d_0 & k = 0 \\ d_{N/2} & k = N/2, \;\; N \text{ even} \\ \pm \left(d_k \cdot d_{N-k}\right)^{1/2} & \text{else} \end{cases}$$

You should run the m-file fourmat in Matlab when you have studied the leaflets on Fourier, Toeplitz and Hankel matrices.