Holger Broman, August 19, 1999

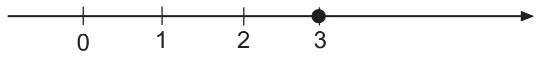
Linear equations, geometry and algebra

Let us start this introduction/refresher by a very simple example, namely the equation

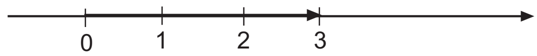
$$2x = 6,$$

where, obviously, x is a real number, that is $x \in \mathbb{R}^1$. What does this equation represent?

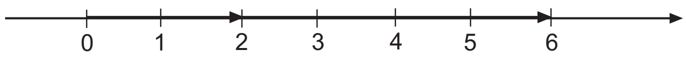
- Algebraically, it means the number 3.
- Geometrically, the solution is a point on the real axis:



  Alternatively, the solution is represented by the vector to this point:



- Geometrically, an alternative interpretation of the equation is: How much of the vector to the point "2" should we take to reach the point "6"?



Things get much more interesting in higher dimensions. Let us nevertheless point out some features. The symbol x, by itself, stands for something that roams in $\mathbb{R}^1$, i.e. it lives with freedom to move in one-dimensional space. The (linear) restriction imposed by an equation such as $2x = 6$ puts x in handcuffs and restricts x to a fixed point, i.e. a zero-dimensional space. Let us carry on this argument in two dimensions.

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Consider

$$\begin{cases} x + 2y &= 4 \\ 3x + 4y &= 10 \end{cases}$$

- Algebraically, we solve this system of equations by using a method named after Gauss, Gaussian elimination[1]. Subtract three times the first equation from the second to obtain the new system

[1] Gaussian elimination was actually used by the Chinese approximately 200 BC.

$$\begin{cases} x + 2y &= 4 \\ -2y &= -2 \end{cases}$$

Now, solve by backtracking:

$$-2y = -2 \quad \Rightarrow \quad y = 1$$

$$y = 1 \quad \Rightarrow \quad x + 2 \cdot 1 = 4 \quad \Rightarrow \quad x = 2,$$

so that the solution is

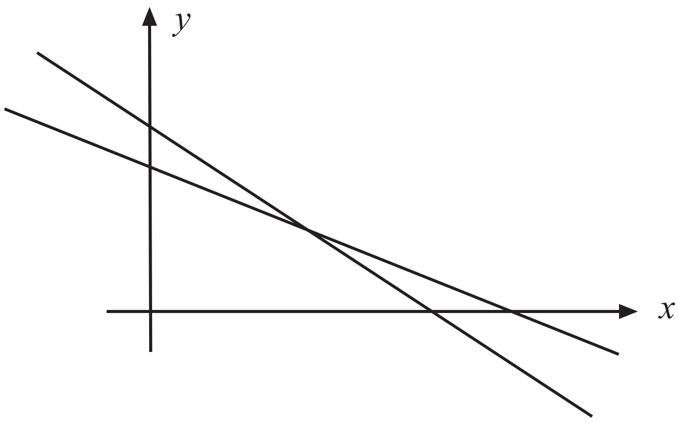
$$\begin{cases} x &= 2 \\ y &= 1 \end{cases}$$

Note 1 Observe the triangular shape of the equation system in the processing.

Question 1 Can you construct a system of equations where the scheme given does not work? Hint: Such systems exist.

- Geometrically, each equation represents a line in the plane (cf. the 1D case, a point on an axis). We repeat the equations:

$$\begin{cases} x + 2y &= 4 \\ 3x + 4y &= 10 \end{cases}$$



The solution is found as the intersection of the two lines.

Note 2 Observe that one equation of two "free" variables reduces the dimensionality from the plane to a line.

Note 3 Expand this to realize that one linear equation in three variables represents a plane in 3D and that a linear equation in N variables represents an $N-1$ dimensional hyperplane.

Question 2 Can you construct a system of equations where there is no intersection?

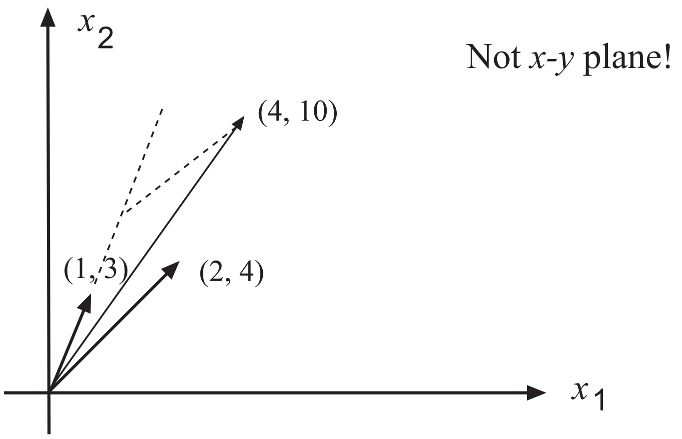
- The alternative geometrical illustration of the system is obtained by rewriting:

$$\left\{\begin{array}{rcl} x+2y &=& 4 \\ 3x+4y &=& 10 \end{array}\right.$$

$$\Longleftrightarrow$$

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} x + \begin{pmatrix} 2 \\ 4 \end{pmatrix} y = \begin{pmatrix} 4 \\ 10 \end{pmatrix},$$

so that the solution is obtained by the answer to the following question: How much (x) of the vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ plus how much (y) of the vector $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ add up to the vector $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$?



Note 4 Of course we find the solution

$$\left\{\begin{array}{rcl} x &=& 2 \\ y &=& 1 \end{array}\right.$$

also in this way.

Note 5 The vectors $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ are linearly independent. This means that the only linear combination of the two vectors that add up to zero,

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} x + \begin{pmatrix} 2 \\ 4 \end{pmatrix} y = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

is given by $x = y = 0$.

Note 6 A linear combination of $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ can be made to reach any point in the plane: The two vectors *span* the plane. As they span the plane and are linearly independent they *form a basis* for the plane.

One often writes this system of equation the following way:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \end{pmatrix}.$$

Question 3 Can you construct a system of equations where the two vectors as used above do not span the plane?

You have probably noticed that the questions 1, 2, and 3 really are the same question, only different viewpoints of the same thing. Here follows, should you need it, exemplifying exercises.

Exercise 1 Solve (in all three ways)

$$\begin{cases} x + 2y &= 4 \\ 2x + 4y &= 10 \end{cases}$$

Exercise 2 Solve (in all three ways)

$$\begin{cases} x + 2y &= 4 \\ 2x + 4y &= 8 \end{cases}$$

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Now, let us generalize all this. Start with the long form of an N by N system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N &= b_1 \\ a_{21}x_1 + \ldots &= b_2 \\ \cdot & \\ \cdot & \\ a_{N1}x_1 + \ldots &= b_N \end{cases}$$

We normally write this:

$$A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & & \vdots \\ a_{N1} & \cdots & a_{NN} \end{bmatrix}$$

$$\mathbf{x} = (x_1 \ x_2 \cdots x_N)^T$$
$$\mathbf{b} = (b_2 \ b_2 \cdots b_N)^T$$

We can also do something intermediate:

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \ldots + \mathbf{a}_N x_N = \mathbf{b},$$

where $\mathbf{a}_1 \ldots$ are the columns of the matrix A. This gives the important aspect of a matrix-vector product:

A matrix-vector product equals a linear combination of the columns of the matrix.

We can now list a set of equivalent statements regarding an N by N matrix A. We will in this list refer to concepts that may not be familiar to you yet. They will be defined in the material to come.

- The columns of A are linearly independent.
- The columns of A span (form a basis of) the N-dimensional space.
- The range space of A is the entire N-dimensional space.
- The matrix A has full rank.
- The determinant of A is not zero.
- No eigenvalue of A equals zero.
- The system of equations $A\mathbf{x} = \mathbf{b}$ has a unique solution in $\mathbf{x}$ for any $\mathbf{b}$.
- The N rows of the equation $A\mathbf{x} = \mathbf{b}$ all represent $N - 1$ dimensional hyperplanes that intersect in exactly one point.
- The rows of A are linearly independent.

Our perception of space and geometry is embodied in algebraic expressions in the coordinates of a given system. As we go on and define a scalar product, this contains the concepts of length and angle.

Please run the m-file lineqsys in Matlab. Display the source code in a separate window. Compare the figures produced with the source code and make sure you understand what is going on. Also, solve the exercises of the program.

```
% lineqsys.m
% Illustration of linear systems of equations
clear

A=[1 2 ; 1 0.5];
b=[6 ; 3];

% The analytical solution
x=A\b

% One row at a time
figure(1), clf, axis equal, hold on
u=[0 3];
%eq1
v=1/A(1,2)*([b(1) b(1)]-A(1,1)*u);
plot(u,v,'r')
%eq2
v=1/A(2,2)*([b(2) b(2)]-A(2,1)*u);
plot(u,v,'g')
% plot analytical solution
plot(x(1),x(2),'b*')
title('The row equations and the solution')

% One column at a time
figure(2), clf, axis equal, hold on
%plot column vectors
c1=A(:,1); c2=A(:,2);
plot([0 c1(1)],[0 c1(2)],'r',c1(1),c1(2),'rp')
plot([0 c2(1)],[0 c2(2)],'g',c2(1),c2(2),'gp')
% Now use the analytical solution
v1=x(1)*c1; v2=x(2)*c2;
plot([0 v1(1)],[0 v1(2)],'r--')
plot([v1(1) v1(1)+v2(1)],[v1(2) v1(2)+v2(2)],'g--')
% plot the location of b
plot([0 b(1)],[0 b(2)],'b',b(1),b(2),'bp')
title('The column vectors and the "b-vector"')

% Make the unique change to a22 to give a system without solution
% Be sure you understand what happens in figures 1 & 2
% Do not worry about the error printout generated by Matlab. There
% is an error, so the printout just shows how good Matlab is.
% With this changed a22, make the unique change to b2 that gives
% a system with infinitely many solutions.
```