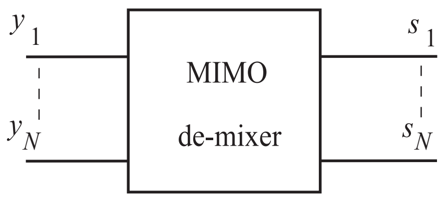
Holger Broman, August 19, 1999

Permutations

Permutation matrices occur now and then in signal processing problems. One example can be found in the blind source separation scenario. The situation is as follows: you have N signal sources that are mixed in a linear MIMO system (Multiple Input Multiple Output):

Now you desire to de-mix the signals y_n, and you use a MIMO de-mixer:



You design the de-mixer so that the outputs s_n each depend on one source signal only! How do you ascertain that s_1 depends on x_1 and so on? The answer is that you cannot always do that, but you can still separate the signals so that

$$s_n = x_k \quad \text{for some } k,$$

Thus, the outputs s_n are permutated versions of the inputs x_n.

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An N by N permutation matrix has N entries that are unity, the rest are zero. Its determinant has unit magnitude.

It follows that each row – and column – has exactly one entry which is unity. It also follows that there are $N!$ distinct permutation matrices of size N by N. Let

$$P = [\mathbf{p}_1 \cdots \mathbf{p}_N],$$

where each column vector has zeros except in one, unique position. Study

$$P^T P = \begin{bmatrix} \mathbf{p}_1^T \\ \vdots \\ \mathbf{p}_N^T \end{bmatrix} [\mathbf{p}_1 \cdots \mathbf{p}_N] = \begin{bmatrix} 1 & & & \\ & \ddots & 0 & \\ & 0 & \ddots & \\ & & & 1 \end{bmatrix} = I.$$

Thus, permutation matrices are norm preserving, and all eigenvalues fall on the unit circle.

The product of two permutation matrices is a permutation matrix. This follows by considering

$$P_1 = [\mathbf{p}_1 \cdots \mathbf{p}_N],$$

and

$$P_2 = [\mathbf{p}_{n1} \cdots \mathbf{p}_{nN}],$$

where P_2 has scrambled the columns of P_1. Now, each row of P_1 has one unity entry that is guaranteed to hit the unity entry of exactly one column of P_2 so that

$$P_1 P_2 = P_3,$$

where P_3 is a permutation matrix.

It seems reasonable that if you keep on permuting the entries of a vector with the same permutation matrix, you will sooner or later return to the original vector. This means that there exists a positive integer K so that

$$P^K \mathbf{x} = \mathbf{x},$$

and that

$$P^K = I.$$

This is indeed true, but we do not give the proof here.

Exercise: List all six permutation matrices of size 3×3 and show the above relationship.

Note: A consequence of $P^K = I$ is that the eigenvalues to P solve the equation $\lambda^K = 1$.

Please run the m-file permutations in Matlab.

```
% permutations.m
clear

% Generate a random permutation matrix
N=5;
v=rand(5,1);
[u,k]=sort(v);
P=zeros(N,N);
for n=1:N
P(n,k(n))=1;
end

% Test what P does to a matrix
A=zeros(N,N);
B=zeros(N,N);
for n=1:N
A(n,:)=(n-1)*N+1:n*N;
B(:,n)=((n-1)*N+1:n*N)';
end

sprintf('Left-multiplication by a permutation matrix equals a row-exchange')
P, A, P*A
pause

sprintf('Right-multiplication by a permutation matrix equals a column-exchange')
B, P, B*P
pause

% Find the periodicity of P
PP=P;
n=1;
while max(max(PP-eye(N))) > 0
n=n+1;
PP=P*PP;
end
sprintf('Power of P that equals the identity')
P, n, P^n
% Can you find the pattern in P that determines the periodicity?
```