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# Vandermonde matrices

Telecommunication systems of today are using not only single antennas, but antenna arrays, and signal processing algorithms have been designed to efficiently make use of the available information. For instance, it is possible to have more than one user (mobile telephones) at any given frequency, using neither time-multiplex nor code-division. The trick is to use spatial diversity, which is possible when using antenna arrays. One particularly simple array is the Uniform Linear Array, the ULA. Modeling the output of such an array with $L$ sensors given $M$ impinging plane waves leads to the use of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j\phi_1} & e^{j\phi_2} & & e^{j\phi_M} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ e^{j(L-1)\phi_1} & e^{j(L-1)\phi_2} & & e^{j(L-1)\phi_M} \end{bmatrix}$$

This matrix is an example of a Vandermonde matrix. Another example is the Fourier matrix already encountered. The general expression is:

$$V = \begin{bmatrix} 1 & \cdots & 1 \\ v_1 & & v_M \\ \vdots & & \vdots \\ v_1^{L-1} & & v_M^{L-1} \end{bmatrix}$$

The special structure of Vandermonde matrices can effectively be used for various purposes. One is to give a proof of conditions for $V$ to have full rank. Assume that there exists a non-trivial vector so that

$$V^H \mathbf{x} = 0.$$

Take the Hermitian transpose

$$\mathbf{x}^H V = 0.$$

Now note that $\mathbf{x}^H V$ is a vector which contains the values of an $L-1$ degree polynomial in the points $v_1, \ldots, v_M$:

$$\mathbf{x}^H V = \left[ P(v_1)\, P(v_2) \cdots P(v_M) \right],$$

where, as an example,

$$P(v_1) = x_0^\star + x_1^\star v_1 + \cdots + x_{L-1}^\star v_1^{L-1}.$$

Now, $P$ can have at most $L-1$ distinct roots, so that if the number of distinct values in the set $v_1, \ldots, v_M$ is larger than $L-1$, then the equation $\mathbf{x}^H V = 0$ has no solutions, and $V$ has full rank. In particular, if $V$ is square, then it is full rank iff all $v_m$ are distinct.

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Another way to make use of the Vandermonde structure, a so-called shift structure, is to observe the following partitionings of $V$:

$$V = \begin{bmatrix} V_1 \\ \text{last row} \end{bmatrix} = \begin{bmatrix} \text{first row} \\ V_2 \end{bmatrix}$$

and the relation

$$V_2 = V_1 \begin{bmatrix} v_1 & & & \\ & \ddots & 0 & \\ & 0 & \ddots & \\ & & & v_M \end{bmatrix}$$

This property is used in the ESPRIT algorithm for estimation of direction-of-arrival of impinging plane waves.