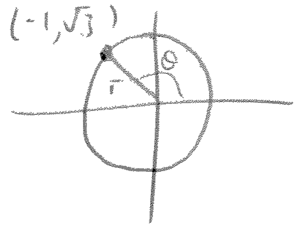
1 (a)

$$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2}$$

$$= \sqrt{1+3} = \sqrt{4} = 2$$

$$\arg(z) = \tan^{-1}\left(\sqrt{3}/_{-1}\right) = \tan^{-1}(-\sqrt{3}) = 2\pi/3$$



$r = 2$
$\theta = 2\pi/3$

Answer : $z = 2e^{2\pi i/3}$

(b)

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 1 & 1 & 1 & 3 \\ 2 & -3 & 1 & -4 \end{array}\right)$$

$R_2 \mapsto 2R_2 - R_1$, $R_3 \mapsto R_3 - R_1$, $R_3 \mapsto 3R_3 + 2R_2$

yields
$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & 2 & 0 \end{array}\right)$$

Back subistitution $\Rightarrow$ $2v = 0 \Rightarrow v = 0$

$3m + v = 6 \Rightarrow m = 2$

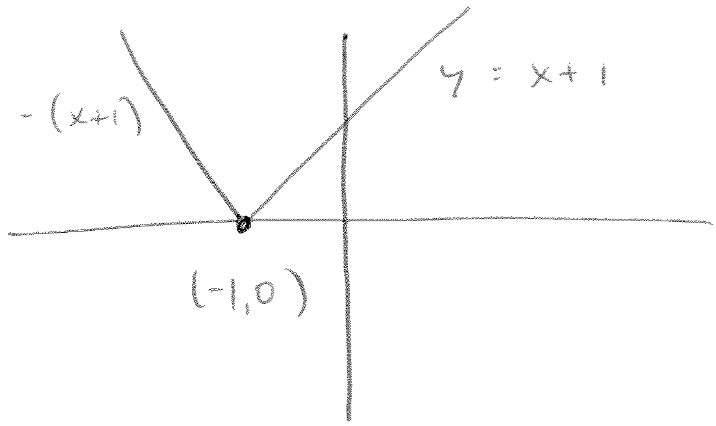
$2t - m + v = 0 \Rightarrow t = 1$

Answer : $t = 1$, $m = 2$, $v = 0$.

(c) Since $f(x)$ is divisible by $(x-3)^{10}$ it will be the case that $f''(x)$ is divisible by $(x-3)^{10-2} = (x-3)^8$. Thus $f''(3)$ must be a multiple of $(3-3)^8 = 0^8 = 0$, i.e.

Ans: $f''(3) = 0$

(d)



Answer: $y = \pm(x+1)$.

(e) (i) Put $t := 1/x$
$x \to \infty \Leftrightarrow t \to 0$, so we have

$$\lim_{t \to 0} \frac{\sin t}{t} = 1, \text{ standard limit}$$

Ans: 1

(ii) The limit does not exist, since if $x \to 1^-$ then $x^2 - 1 < 0$ and $\ln(x^2-1)$ is undefined. If the limit had been as $x \to 1^+$ then we would instead have the following :

$$\lim_{x\to 1^+} \ln(x^2-x) - \ln(x^2-1)$$

$$= \lim_{x\to 1^+} \ln\left(\frac{x^2-x}{x^2-1}\right)$$

$$= \lim_{x\to 1^+} \ln\left(\frac{x\cancel{(x-1)}}{(x+1)\cancel{(x-1)}}\right)$$

$$= \lim_{x\to 1^+} \ln\left(\frac{x}{x+1}\right) \underset{(x:=1)}{=} \ln(1/2)$$

$$= -\ln 2 . \quad \underline{\text{Ans}} \; -\ln 2 .$$

(7) $2 = \tan\theta \sin\theta = \dfrac{\sin\theta}{\cos\theta} \sin\theta$

$$= \frac{\sin^2\theta}{\cos\theta} = \frac{1-\cos^2\theta}{\cos\theta} .$$

let $x := \cos\theta$, thus $2 = \dfrac{1-x^2}{x} \Rightarrow$

$x^2+2x-1=0 \Rightarrow x = -1 \pm \sqrt{2}$.

But $\cos\theta \in [-1,1]$ a priori, so $-1-\sqrt{2}$ is not a valid solution.

Ans $\cos\theta = -1+\sqrt{2}$

2 a

let $\vec{v} := (3,-2,4)$

$\vec{w} := (1,2,0)$

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 3 & -2 & 4 \\ 1 & 2 & 0 \end{vmatrix}$$

$$= i(-2\cdot 0 - 2\cdot 4) - j(3\cdot 0 - 1\cdot 4) + k(3\cdot 2 - 1\cdot(-2))$$

$$= -8\bar{i} + 4\bar{j} + 8\bar{k}$$

This vector is $\perp$ both $\vec{v}$ & $\vec{w}$.

Normalising gives a unit vector:

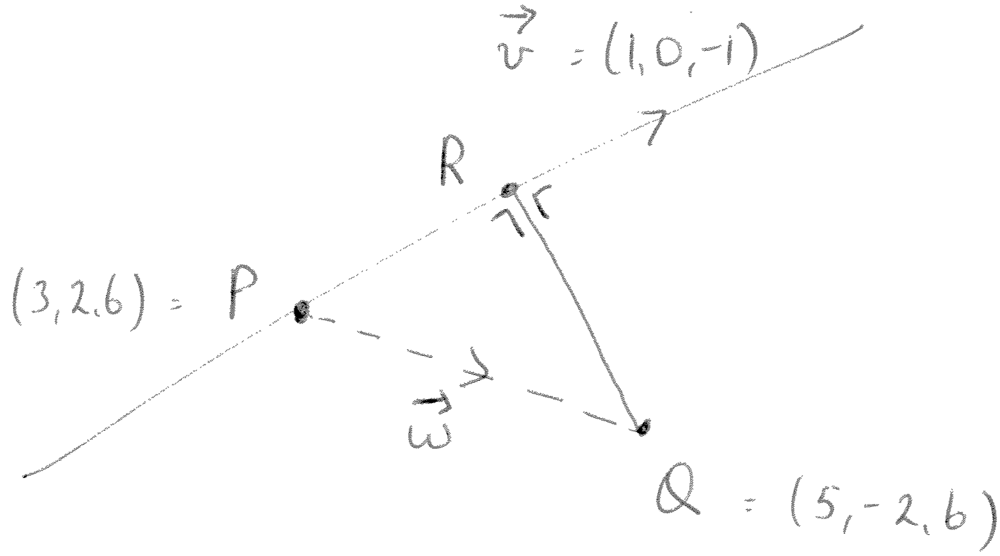
$$\frac{-8\bar{i} + 4\bar{j} + 8\bar{k}}{\sqrt{8^2 + 4^2 + 8^2}}$$

$$= \frac{4(-2\bar{i} + \bar{j} + 2\bar{k})}{4\sqrt{2^2 + 1^2 + 2^2}}$$

$$= \frac{-2i + j + k}{\sqrt{9}}$$

$$= \frac{-2i + j + k}{3}$$: Answer

(b)



$$R = P + \operatorname{proj}_{\vec{w}} \vec{v}$$

$$= P + \left(\frac{\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right) \vec{v}$$

$$\vec{w} = Q - P = (5,-2,6) - (3,2,6) = (2,-4,0)$$

$$\vec{w} \cdot \vec{v} = (2,-4,0) \cdot (1,0,-1) = 2$$

$$\vec{v} \cdot \vec{v} = (1,0,-1) \cdot (1,0,-1) = 2$$

$$\Rightarrow \quad R = (3,2,6) + \frac{2}{2}(1,0,-1)$$

$$= (4,2,5)$$: Answer

(3) $p(x)$ växande if $p'(x) > 0$.

$$p'(x) = 12x^3 - 24x^2 - 60x + 72$$
$$= 12(x^3 - 2x^2 - 5x + 6)$$

So we need to know where

$$x^3 - 2x^2 - 5x + 6 > 0$$

To do so we solve

$$x^3 - 2x^2 - 5x + 6 = 0.$$

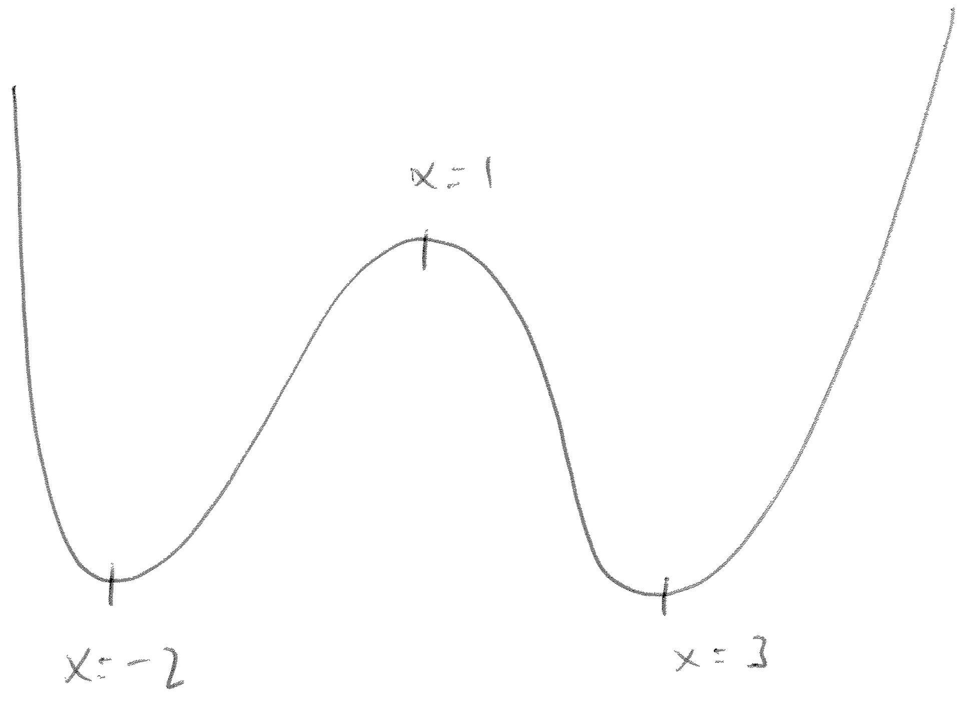
By inspection we see that $x = 1$ is a root.

$$
\begin{array}{r|l}
 & x^2 - x - 6 \\
\hline
x - 1 & x^3 - 2x^2 - 5x + 6 \\
 & x^3 - x^2 \\
\hline
 & -x^2 - 5x \\
 & -x^2 + x \\
\hline
 & -6x + 6
\end{array}
$$

$$x^2 - x - 6 = 0 \iff (x-3)(x+2) = 0$$
$$\iff x = 3 \text{ or } x = -2.$$

Hence the graph of $p(x)$ has critical points at $x = -2$, $x = 1$, $x = 3$.

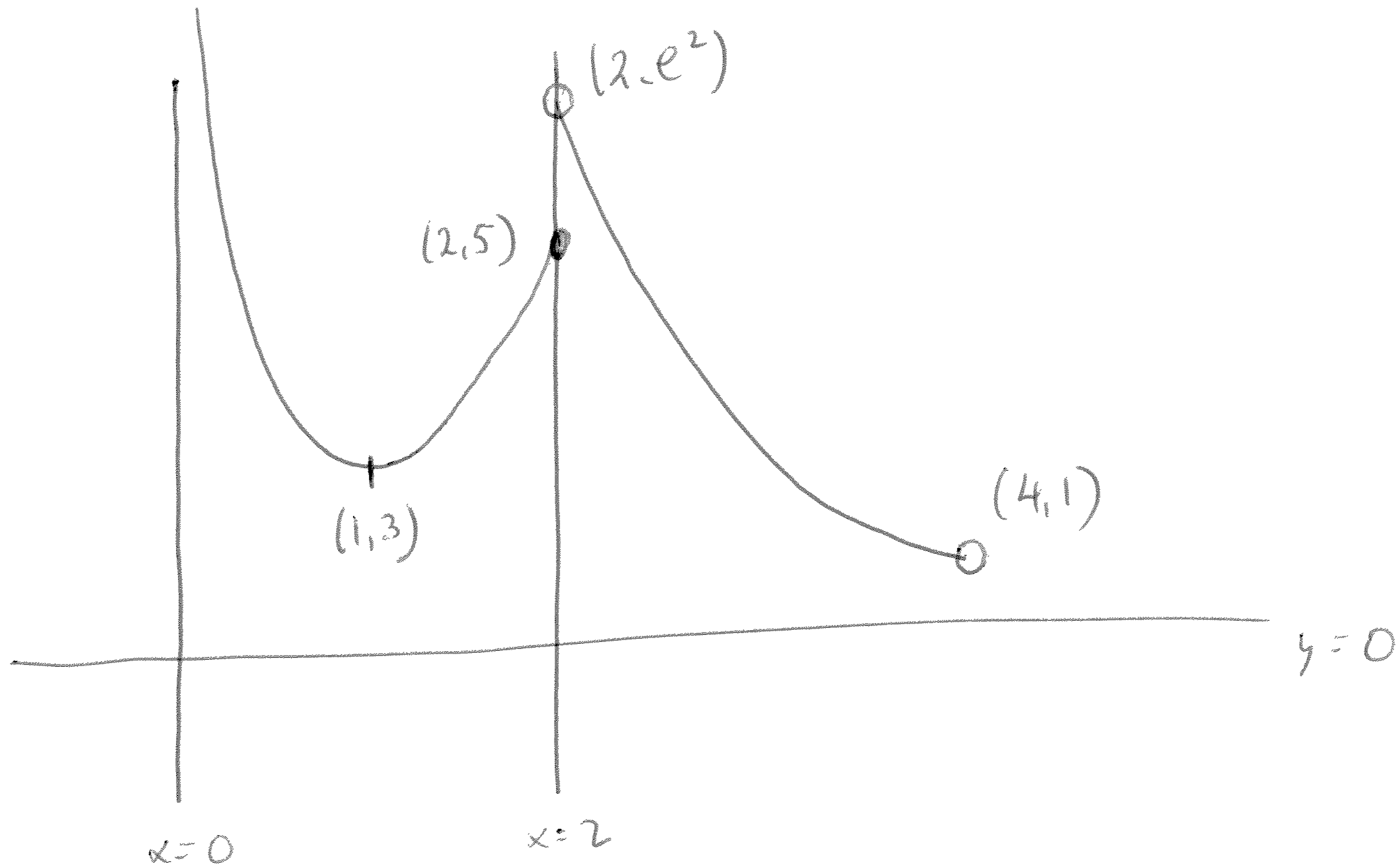
Since $p(x)$ is a 4^{th} degree polynomial its graph therefore looks something like this:



Hence, $p(x)$ is increasing when $x \in (-2, 1)$ or $x > 3$.

<u>Answer</u> $x \in (-2,1) \cup (3, \infty)$.

(4) The graph looks as follows:



<u>Note</u>: The local minimum at (1,3) is found by differentiating $x^2 + 2/x$.

(5) First we find $M(a)$ for a fixed $a \in \mathbb{R}$ by finding the local min. of the function $f(x)$, considered as a quadratic function of x only.

$$f'(x) = 2x - 2a = 0 \text{ @ } x = a$$

$$\Rightarrow M(a) = f(a) = a^2 - 2a^2 + 2a^2 - a = a^2 - a$$

Step 2 is then to minimise $M(a)$, for $a \in \mathbb{R}$.

$M'(a) = 2a - 1 = 0$ @ $a = 1/2$

Thus $\min M(a) = (1/2)^2 - 1/2 = -1/4$

Ans: The minimum value is $-1/4$ & is attained at $a = 1/2$.

6 (a) False. $\ln x < 0$ when $0 < x < 1$.

(b) False. $\frac{d}{dx}(e^{x \sin x}) =$

$$e^{x \sin x} \cdot \frac{d}{dx}(x \sin x)$$

$$= e^{x \sin x}(x \cos x + \sin x)$$

(c) True, it will have the same periodic behaviour.

(d) False. $1/2 \neq (1/2)^2 = 1^2/2^2 = 1/4$.

(e) True. This is precisely the definition of $\lim_{x \to 0} f(x) = 0$. See Section 1.5 in

(f) False. Rather the reverse applies (see Theorem 2.3.1)

7) (a) Along with

(i) $u \cdot (v \times w) = v \cdot (w \times u)$

we'll also use

(ii) $a \times b = - b \times a$.

(iii) $u \cdot (v + w) = u \cdot v + u \cdot w$

We need to show that

$$u \times (v + w) = (u \times v) + (u \times w)$$

To do so it suffices to show that the two sides have the same scalar product with any vector. So let ξ be an arbitrary vector. On the left, we have

$$\xi \cdot u \times (v + w)$$

$$\overset{(ii)}{=} \xi \cdot [-(v+w) \times u]$$

$$- \xi \cdot [(v+w) \times u]$$

$$\overset{(i)}{=} -(v+w)\cdot[u\times\xi]$$

$$\overset{(iii)}{=} -v\cdot(u\times\xi) - w\cdot(u\times\xi)$$

$$\overset{(i)}{=} -\xi\cdot(v\times u) - \xi\cdot(w\times u)$$

$$\overset{(ii)}{=} \xi\cdot(u\times v) + \xi\cdot(u\times w)$$

$$\overset{(iii)}{=} \xi\cdot[(u\times v)+(u\times w)],$$

which is what we have on the right.

Q.E.D.

(b) Let $\varepsilon>0$ be given. We must produce $\partial>0$ s.t.

$$0<|x-a|<\partial \Rightarrow |f(x)g(x)|<\varepsilon.$$

Since $\lim\limits_{x\to 0} f(x)=0$ there exists $\partial^*>0$ such that

$$0<|x-a|<\partial^* \Rightarrow |f(x)|<\varepsilon/M.$$

Also, by the assumption on $g(x)$, there exists $\delta^{**} > 0$ s.t.

$$0 < |x-a| < \delta^{**} \Rightarrow |g(x)| < M.$$

Set $\delta := \min\{\delta^*, \delta^{**}\}$.

Then $0 < |x-a| < \delta \Rightarrow$

(i) $|f(x)| < \varepsilon/M$

(ii) $|g(x)| < M$

$$\Rightarrow |f(x)g(x)| = |f(x)||g(x)| < \varepsilon/M \cdot M = \varepsilon$$

i.e:

$$0 < |x-a| < \delta \Rightarrow |f(x)g(x)| < \varepsilon,$$

Q.E.D.