MATEMATIK
Chalmers

Tentamen i Inledande matematik för Z1, (TMV120)
Tid: 2008-10-24, kl 08.30 - 12.30
Hjälpmedel: Inga
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Skriv tentamenskoden på varje inlämnat blad.
Betygsgränser: 20 - 29 p ger betyget 3, 30 - 39 p ger betyget 4 och 40 eller mer betyget 5.
(Bonuspoäng från duggor hösten 2008 inkluderas.)
Lösningar läggs ut på kursens webbsida tidigast fredag 24/10 em.
Resultat meddelas via Ladok senast ca. tre veckor efter tentamenstillfället.

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1. Till denna uppgift ska du **endast lämna in svar**, alltså utan motiveringar.

   a) Lös ekvationssystemet (2p)

$$\begin{cases} 2x + y + z = 3 \\ x + 2y - z = 2 \\ 2x - 5y + 7z = 1 \end{cases}$$

   b) Lös ekvationen $z^4 = -1$ fullständigt. (3p)

   c) I vilket eller vilka intervall av x-värden är funktionen $f(x) = xe^{-x}$ konkav ner (=concave down)? (2p)

   d) Beräkna följande gränsvärden: (3p)

   i. $\lim_{x\to\infty} \frac{x^2 \ln x + x^3}{(\ln x)^6 + 2x^3}$ ii. $\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{\frac{\pi}{2} - x}$ iii. $\lim_{x\to 0}(1+5x)^{4/x}$

   e) Funktionen $f(x) = x + \log_2 x$ är injektiv då $x > 0$. Låt $g(x)$ vara dess invers. Beräkna $g'(3)$. (Deriveringsregel: $\frac{d}{dx}\log_2 x = \frac{1}{x \ln 2}$). (2p)

   f) I en rätvinklig triangel ökar kateternas längder med hastigheterna 3 m/s respektive 4 m/s. Hur snabbt växer triangelns area i det ögonblick då kateterna är 3 m resp. 4 m långa? (2p)

Till uppgifterna 2-5 ska du lämna in fullständiga lösningar.

2. Givet är de tre punkterna $A = (1,1,1)$, $B = (7,3,-2)$ och $C = (4,9,-4)$. Låt L vara linjen genom A och B. Låt P vara planet som innehåller A, B och C. Låt Q vara planet $x - y - 2z = -12$. (6p)

   a) Ange en ekvation för L.

   b) Ange en ekvation för P.

   c) Beräkna skärningspunkten mellan L och Q.

Var god vänd!

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3. Ange det största och det minsta värdet som antas av funktionen $f(x) = e^{-x^2}(x + \frac{1}{2})$ på intervallet $[0, 10]$. (6p)

4. Rita grafen till funktionen $f(x) = \dfrac{\sqrt{x^2+2}}{x-2}$. (6p)
   Ange eventuella lokala extrempunkter och asymptoter.
   (Konvexitet/konkavitet behöver inte utredas.)

5. En triangel har sina hörn i punkterna $(0, 0)$, $(2, 0)$ och $(1, 2)$. Låt A vara punkten $(1, 0)$. Om en linje parallell med x-axeln dras någonstans mellan $y = 0$ och $y = 2$, så kommer den att skära två av triangelns sidor, säg i punkterna B och C. På vilken höjd ska man dra denna linje för att minimera omkretsen av triangeln ABC? (6p)

6. Avgör vilka av följande påståenden som är sanna respektive falska. Du behöver inte motivera dig. Rätt svar ger 1p, inget svar 0p och fel svar -1p. Dock ej mindre än 0p totalt. (6p)

   a) För varje komplext tal z gäller att $|z| = |\text{Re}(z)| + |\text{Im}(z)|$.
   b) Om ett linjärt ekvationssystem har fler ekvationer än obekanta så kan systemet inte ha entydig lösning.
   c) Olikheten $e^x \geq x + 1$ är sann för alla reella tal x.
   d) Låt $\boldsymbol{u}$ vara en vektor i rummet. Om $\boldsymbol{u} \cdot \boldsymbol{u} = 0$ så måste $\boldsymbol{u} = \mathbf{0}$.
   e) Om $f(x)$ är definierad i hela $\mathbb{R}$ och deriverbar, och har totalt 2008 olika nollställen, så har ekvationen $f'(x) = 0$ fler än 1000 lösningar.
   f) Om $f \circ f$ är en injektiv (one-to-one) funktion, så måste f själv vara injektiv.

7. a) Definiera *derivatan* av en funktion f i en punkt x. (2p)
   b) Bevisa att $\frac{d}{dx}(\sin x) = \cos x$. Du behöver inte bevisa eventuella hjälpsatser om trigonometri eller gränsvärden. (4p)

Be'hatzlacha!
/Peter

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Lösningar

1.(a) In matrix form the system is

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ 1 & 2 & -1 & 2 \\ 2 & -5 & 7 & 1 \end{array}\right).$$

The sequence of row operations

$$R_2 \mapsto 2R_2 - R_1, \quad R_3 \mapsto R_3 - R_1, \quad R_3 \mapsto R_3 + 2R_2,$$

transforms the system to the echelon form

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ 0 & 3 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus we can choose z as a free variable. Back substitution gives, first of all,

$$3y - 3z = 1 \Rightarrow y = \frac{1}{3} + z,$$

and then

$$2x + y + z = 3 \Rightarrow 2x + (\frac{1}{3} + z) + z = 3 \Rightarrow x = \frac{4}{3} - z.$$

ANSWER : $\{(\frac{4}{3} - z, \frac{1}{3} + z, z) : z \in \mathbb{R}\}$.

(b) Put $z = r(\cos\theta + i\sin\theta)$, thus $z^4 = r^4(\cos 4\theta + i\sin 4\theta)$. Note that, in polar form, $-1 = 1(\cos\pi + i\sin\pi)$. Thus, if $z^4 = -1$, it follows that

$$r^4 = 1, \quad 4\theta = \pi + 2\pi n, \text{ for some } n \in \mathbb{Z}.$$

So $r = 1$ and we get four distinct solutions by taking $n = 0, 1, 2, 3$ respectively. The corresponding values of θ are $\pi/4, 3\pi/4, 5\pi/4$ and $7\pi/4$ respectively. Thus the four solutions to the equation are, in polar form,

$$\begin{aligned} z_1 &= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}, \\ z_2 &= \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}, \\ z_3 &= \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}, \\ z_4 &= \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}. \end{aligned}$$

Noting that $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, we can write the solutions as

$$z_{1,3} = \pm\frac{1}{\sqrt{2}}(1+i),$$
$$z_{2,4} = \overline{z_{1,3}} = \pm\frac{1}{\sqrt{2}}(1-i).$$

(c) f is concave down when $f'' < 0$. We compute in turn (using the Leibniz product rule)

$$f'(x) = e^{-x}(1-x), \qquad f''(x) = e^{-x}(x-2).$$

Thus $f''(x) < 0 \Leftrightarrow e^{-x}(x-2) < 0 \Leftrightarrow x-2 < 0 \Leftrightarrow x < 2$.
ANSWER : $(-\infty, 2)$.

(d) **i** Since $\frac{\ln x}{x} \to 0$ as $x \to \infty$, the dominant terms in the numerator and denominator are x^3 and $2x^3$ respectively. Thus the limit will be $1/2$.
ii Let $y := \frac{\pi}{2} - x$. Then, in terms of y, the limit becomes

$$\lim_{y\to 0}\frac{\cos(\frac{\pi}{2}-y)}{y} = \lim_{y\to 0}\frac{\sin y}{y} = 1.$$

iii Let $y := 1/x$, so in terms of y the limit becomes

$$\lim_{y\to\infty}\left(1+\frac{5}{y}\right)^{4y} = \lim_{y\to\infty}\left[\left(1+\frac{5}{y}\right)^{y}\right]^4 = (e^5)^4 = e^{20}.$$

(e) We use the formula

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))},$$

which, since we're using the notation g instead of f^{-1} here, we can rewrite as

$$g'(x) = \frac{1}{f'(g(x))}.$$

Setting $x = 3$ we have the formula

$$g'(3) = \frac{1}{f'(g(3))}.$$

First, we have that $g(3)$ is the solution of the equation $f(x) = 3$, i.e.: the solution of

$$x + \log_2(x) = 3.$$

One sees directly that the solution is $x = 2$. Thus $g'(3) = 1/f'(2)$. Using the given differentiation rule we can assert that $f'(x) = 1 + \frac{1}{x\ln 2}$. Finally, then, inserting $x = 2$ yields

$$g'(3) = \frac{1}{1+\frac{1}{2\ln 2}} = \frac{2\ln 2}{2\ln 2 + 1}.$$

(f) Let x and y denote the lengths of the respective catheters and A the area of the triangle. Thus

$$A = \frac{1}{2}xy \tag{1}$$

and the information given is that

$$x = 3, \quad y = 4, \quad \frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 4. \tag{2}$$

Differentiating (1) w.r.t. t and inserting the information in (2) we find that

$$\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right) = \frac{1}{2}(3 \cdot 4 + 4 \cdot 3) = 12.$$

ANSWER : The area is increasing at a rate of 12 m/s at the given instant.

2. (a) Riktningen för L ges av $\boldsymbol{v}_1 = (7, 3, -2) - (1, 1, 1) = (6, 2, -3)$. Linjen ges därmed i skalärparameterform av

$$L = \{((1 + 6t, 1 + 2t, 1 - 3t) : t \in \mathbb{R}\}.$$

(b) Vi har $\vec{AB} = (6, 2, -3)$ och $\vec{AC} = (4, 9, -4) - (1, 1, 1) = (3, 8, -5)$. En normal till planet ges av

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 6 & 2 & -3 \\ 3 & 8 & -5 \end{vmatrix} = (2(-5) - 8(-3), 3(-3) - 6(-5), 6(8) - 3(2)) = (14, 21, 42).$$

Taking A as the base point, the equation for the plane can thus be written as

$$14(x - 1) + 21(y - 1) + 42(z - 1) = 0,$$

which simplifies nicely to $2x + 3y + 6z = 11$.

(c) We insert the parametric equation for L into that for Q and solve for t. The resulting equation for t is

$$(1 + 6t) - (1 + 2t) - 2(1 - 3t) = -12,$$

whose solution is $t = -1$. Thus the point of intersection of L and Q is $(1 + 6(-1), 1 + 2(-1), 1 - 3(-1)) = (-5, -1, 4)$.

3. We have to examine the endpoints and the critical points inside the interval. For the latter, we compute (using the product rule)

$$f'(x) = e^{-x^2}(1 - x - 2x^2).$$

Thus $f'(x) = 0 \Leftrightarrow 1 - x - 2x^2 = 0 \Leftrightarrow 2x^2 + x - 1 = 0 \Leftrightarrow (2x-1)(x+1) = 0 \Leftrightarrow x = 1/2$ or $x = -1$. The latter point lies outside the interval of interest, so we ignore it. Thus we have to compare $x = 1/2$ with the endpoints $x = 0$ and $x = 10$. We have

$$f(0) = 1/2, \quad f(1/2) = e^{-1/4}, \quad f(10) = \frac{21}{2e^{100}}.$$

Obviously, the third value is the smallest of these, and one easily checks that the second one is the largest (since $e^{1/4} < 2$, or equivalently, $e < 16$).
ANSWER : The largest value is $e^{-1/4}$ and the smallest is $\frac{21}{2e^{100}}$.

4. *Step 0* : Search for any obvious symmetries.

There aren't any.

Step 1 : Investigate behaviour as $x \to \pm\infty$.

Noting that the square root is always positive we have that

$$\lim_{x\to+\infty} f(x) = +1, \qquad \lim_{x\to-\infty} f(x) = -1.$$

Thus the lines $y = \pm 1$ are horizontal asymptotes.

Step 2 : Investigate the domain and possible vertical asymptotes.

There will be a vertical asymptote at $x = 2$ and we check that

$$\lim_{x\to 2^+} f(x) = +\infty, \qquad \lim_{x\to 2^-} f(x) = -\infty.$$

Step 3 : Find and classify the critical points.

Using the quotient rule, we compute

$$f'(x) = \frac{(x-2)\frac{x}{\sqrt{x^2+2}} - \sqrt{x^2+2}}{(x-2)^2}.$$

The quotient is zero when the numerator is, so we must solve the equation

$$(x-2)\frac{x}{\sqrt{x^2+2}} - \sqrt{x^2+2} = 0.$$

Multiplying through by $\sqrt{x^2+2}$ simplifies this to

$$x(x-2) - (x^2+2) = 0 \Rightarrow -2x - 2 = 0 \Rightarrow x = -1.$$

Note that $f(-1) = -1/\sqrt{3} > -1$. Thus, from the information gleaned in Steps 1 and 2, we know that the critical point at $x = -1$ must be a local maximum.

Note that $f(x) = 0$ has no solutions, since the numerator cannot be zero. We can now draw the graph. Check your email for a picture.

5. Note that $|AB| = |AC|$ no matter where we draw the horizontal line. Suppose the horizontal line is drawn at $y = 2t$, so $t \in [0,1]$. Then the points B and C will be $(t, 2t)$ and $(2-t, 2t)$ respectively. Then the perimeter of the triangle, as a function of t, will be given by

$$P(t) = |AB| + |AC| + |BC| = 2|AB| + |BC|$$
$$= 2\sqrt{(t-1)^2 + (2t)^2} + (2-2t) = 2(\sqrt{5t^2 - 2t + 1} + 1 - t).$$

This is the function we want to minimise. So we compute

$$P'(t) = 2 \cdot \left(\frac{5t-1}{\sqrt{5t^2 - 2t + 1}} - 1 \right).$$

Setting $P'(t) = 0$ yields the equation

$$\frac{5t-1}{\sqrt{5t^2 - 2t + 1}} - 1 = 0,$$

hence

$$(5t-1)^2 = 5t^2 - 2t + 1 \Rightarrow 5t^2 - 2t = 0 \Rightarrow t(5t-2) = 0,$$

so $t = 0$ or $t = 2/5$. Now check that $P(0) = 6$ and $P(2/5) = 16/5$. we conclude that the minimum perimeter is obtained for $t = 2/5$, i.e.: when we draw the horizontal line at $y = 4/5$.

6.(a) False. Instead $|z| = \sqrt{\text{Re}(z)^2 + \text{Im}(z)^2}$.

(b) False. If some equations are redundant, we could be left with a system with the same number of irredundant equations as unknowns, which will usually have a unique solution.

(c) True. See Theorem x in Section 3.y.

(d) True. $\boldsymbol{u} \cdot \boldsymbol{u} = ||\boldsymbol{u}||^2$, and only the zero vector has zero length.

(e) True. By Rolle's theorem there must be at least one critical point between any pair of zeroes. Thus f has, in fact, at least 2007 critical points.

(f) True. If $f(x_1) = f(x_2)$ then also $(f \circ f)(x_1) = f(f(x_1)) = f(f(x_2)) = (f \circ f)(x_2)$.

7.(a)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

(b) See Section 2.5 in Adams or your lecture notes for a proof, where the following three facts are used :

(i) $\lim_{x \to 0} \frac{\sin x}{x} = 1$,
(ii) $\sin(A + B) = \sin A \cos B + \cos A \sin B$,
(iii) $1 - \cos A = 2 \sin^2 \frac{A}{2}$.