Anonym kod	TMV122/177 Inledande Matematik Z/TD 2015-08-26	Poäng

1. Till nedanstående uppgifter skall korta lösningar redovisas, samt svar anges, på anvisad plats (endast lösningar och svar på detta blad, och på anvisad plats, beaktas).

(a) Skriv talet $(3 + \sqrt{3}i)^{258}$ på formen $a + ib$. (2 p)

Lösning: $z = 3+\sqrt{3}\,i \Rightarrow |z| = \sqrt{9+3} = \sqrt{12}$

$\arg(z) = \arctan\left(\frac{\sqrt{3}}{3}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\Rightarrow z^{258} = 12^{258/2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{258} = \{\text{de Moivre}\} =$

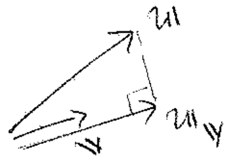
$= 12^{129}\left(\cos\left(\frac{258\pi}{6}\right) + i\sin\left(\frac{258\pi}{6}\right)\right) = 12^{129}\left(\cos(43\pi) + i\sin(43\pi)\right) =$

$= 12^{129}\left(\cos(\pi) + i\sin(\pi)\right) = \underline{-12^{129}}$

Svar: ....

(b) Beräkna ortogonalprojektionen av $\mathbf{u} = (7, -2, -2)$ på $\mathbf{v} = (1, 0, 1)$. (2 p)

Lösning:



$u_v = \frac{u \cdot v}{|v|^2}\, v =$

$= \frac{7-2}{2}(1,0,1) = \left(\frac{5}{2}, 0, \frac{5}{2}\right)$

Svar: $\left(\frac{5}{2}, 0, \frac{5}{2}\right)$

(c) Bestäm normallinjen till kurvan (2 p)

$$2x + y - \sqrt{2}\sin(xy) = \pi/2$$

i punkten $(\pi/4, 1)$.

Lösning: Derivera m.a.p. x : $2 + y' - \sqrt{2}\cos(xy)(y + xy') = 0$

Stoppa in $\left(\frac{\pi}{4}, 1\right)$: $2 + y' - \sqrt{2}\cdot\frac{1}{\sqrt{2}}\left(1 + \frac{\pi}{4}y'\right) = 0$

$\Leftrightarrow \frac{\pi}{4}y' - y' = 1 \Leftrightarrow y'\big|_{(\frac{\pi}{4}, 1)} = \frac{4}{\pi - 4} = k_T$

$k_T \cdot k_N = -1 \Leftrightarrow k_N = -\frac{1}{k_T} = \frac{4-\pi}{4}$

Svar: $y = \frac{4-\pi}{4}\left(x - \frac{\pi}{4}\right) + 1$

(d) Funktionen $f(x) = \ln\sqrt{1+x^3}$ är inverterbar då $x > -1$. Bestäm $(f^{-1})'(\ln 3)$. (2 p)

Lösning: $f(2) = \ln\sqrt{1+8} = \ln 3 \Rightarrow f^{-1}(\ln 3) = 2$

$$f'(x) = \frac{1}{\sqrt{1+x^3}} \cdot \frac{1}{2} \cdot \frac{3x^2}{\sqrt{1+x^3}} = \frac{3x^2}{2(1+x^3)}$$

$$(f^{-1})'(\ln 3) = \frac{1}{f'(f^{-1}(\ln 3))} = \frac{1}{f'(2)} = \frac{2(1+8)}{3 \cdot 4} =$$

$$= \frac{2 \cdot 9}{3 \cdot 4} = \frac{3}{2}$$

Svar: 3/2

(e) Beräkna gränsvärdena (1+1 p)

(i) $\lim\limits_{x \to -\infty} (\sqrt{x^2+x} + x)$

(ii) $\lim\limits_{x \to 0} \dfrac{\arcsin(x)}{\sin(x)}$

Lösning: (i) $\lim\limits_{x \to -\infty} \sqrt{x^2+x} + x = \lim\limits_{x \to -\infty} \dfrac{(\sqrt{x^2+x}+x)(\sqrt{x^2+x}-x)}{\sqrt{x^2+x}-x} =$

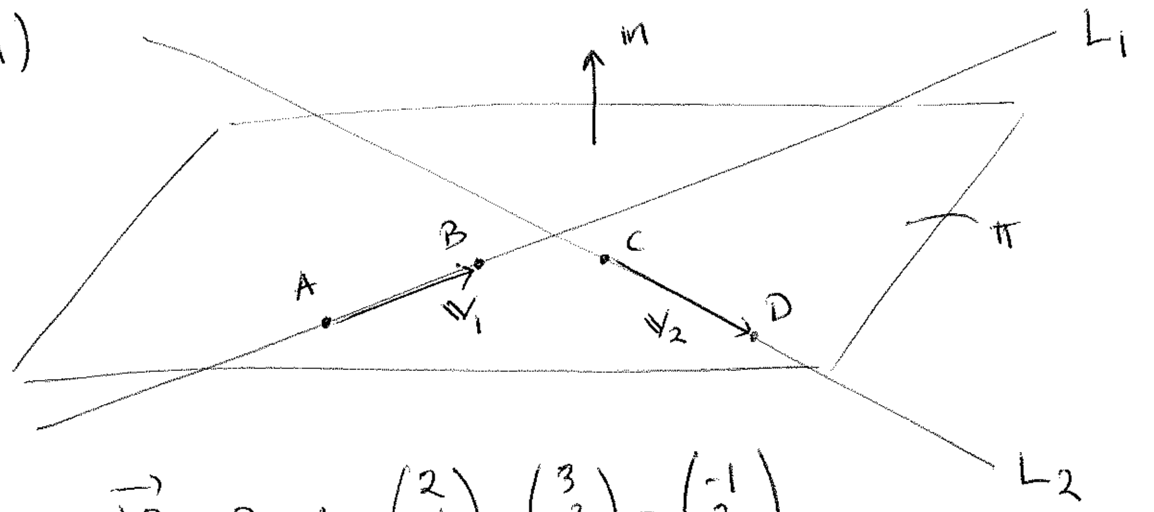
$$= \lim_{x \to -\infty} \frac{\cancel{x^2} + x - \cancel{x^2}}{\sqrt{x^2+x} - x} = \lim_{x \to -\infty} \frac{x}{|x|\sqrt{1+\frac{1}{x}} - x} = \left\{ \begin{array}{l} x < 0 \Rightarrow \\ \Rightarrow |x| = -x \end{array} \right\} =$$

$$= \lim_{x \to -\infty} \frac{\cancel{x}}{-\cancel{x}\left(\sqrt{1+\frac{1}{x}} + 1\right)} = \lim_{x \to -\infty} -\frac{1}{\sqrt{1+\frac{1}{x}}+1} = -\frac{1}{2}$$

(ii) $\lim\limits_{x \to 0} \dfrac{\arcsin(x)}{\sin(x)} = \{\text{l'Hôpital}\} = \lim\limits_{x \to 0} \dfrac{\frac{1}{\sqrt{1+x^2}}}{\cos(x)} = \dfrac{1}{1} = 1$

Svar: (i) $-1/2$ (ii) 1

2. (a)



$$\mathbf{v}_1 = \overrightarrow{AB} = B - A = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

$$\mathbf{v}_2 = \overrightarrow{CD} = D - C = \begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ 6 \\ -2 \end{pmatrix}$$

$$\mathbf{n} \perp \mathbf{v}_1 \;\&\; \mathbf{n} \perp \mathbf{v}_2 \Rightarrow$$

$$\Rightarrow \mathbf{n} \,/\!/\, \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -1 \\ -6 & 6 & -2 \end{vmatrix} = (-6, 4, 6)$$

$$\Rightarrow \{\text{längden av } \mathbf{n} \text{ irrelevant}\} \Rightarrow \mathbf{n} = \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix}$$

$$\pi : -3x + 2y + 3z = \tilde{D}$$

$$C \in \pi \Rightarrow \tilde{D} = -3 \cdot 3 + 2 \cdot 0 + 3 \cdot 2 = -3$$

$$\therefore \pi : -3x + 2y + 3z = -3$$

(b) $L_1 : \begin{cases} x = 3 - t \\ y = -3 + 2t \\ z = 4 - t \end{cases}, \ t \in \mathbb{R}, \quad L_2 : \begin{cases} x = 3 - 6s \\ y = 6s \\ z = 2 - 2s \end{cases}, \ s \in \mathbb{R}$

Skärningspunkt om för något s, t

$$\begin{cases} 3 - t = 3 - 6s \\ -3 + 2t = 6s \\ 4 - t = 2 - 2s \end{cases} \iff \begin{cases} t - 6s = 0 \\ 2t - 6s = 3 \\ t - 2s = 2 \end{cases} \Rightarrow$$

$$\Rightarrow \left(\begin{array}{cc|c} 1 & -6 & 0 \\ 2 & -6 & 3 \\ 1 & -2 & 2 \end{array}\right) \begin{matrix} -2 & -1 \end{matrix} \sim \begin{pmatrix} 1 & -6 & 0 \\ 0 & 6 & 3 \\ 0 & 4 & 2 \end{pmatrix} \begin{matrix} 1 & -\frac{4}{6} \end{matrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 6 & 3 \\ 0 & 0 & 0 \end{pmatrix} \cdot \tfrac{1}{6} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} t = 3 \\ s = 1/2 \end{cases}$$

$t = 3$ insatt i L_1: $\begin{cases} x = 3 - 3 = 0 \\ y = -3 + 2 \cdot 3 = 3 \\ z = 4 - 3 = 1 \end{cases}$

Kontroll: $s = 1/2$ insatt i L_2: $\begin{cases} x = 3 - 6 \cdot \frac{1}{2} = 0 \\ y = 6 \cdot \frac{1}{2} = 3 \\ z = 2 - 2 \cdot \frac{1}{2} = 1 \end{cases}$ ok!

<u>Svar:</u> $(0, 3, 1)$

3. $f(x) = \frac{1}{2x} - \frac{2}{\sqrt{x}}$

Steg 1 : $D_f = (0, \infty)$

Steg 2 : $f'(x) = -\frac{1}{2x^2} + \frac{1}{x^{3/2}}$

$f'(x) = 0 \Leftrightarrow \frac{1}{x^{3/2}} = \frac{1}{2x^2} \Leftrightarrow x^{1/2} = \frac{1}{2} \Rightarrow \underline{\underline{x = \frac{1}{4}}}$

Steg 3 : $f''(x) = \frac{1}{x^3} - \frac{3}{2x^{5/2}}$

$f''(x) = 0 \Leftrightarrow \frac{3}{2x^{5/2}} = \frac{1}{x^3} \Leftrightarrow x^{1/2} = \frac{2}{3} \Rightarrow \underline{\underline{x = \frac{4}{9}}}$

Steg 4 :

		$1/4$		$4/9$	
f'	$- - -$	0	$+ + +$		$+ + +$
f''	$+ + +$		$+ + +$	0	$- - -$
f	$\searrow \cup$	-2	$\nearrow \cup$	$-\frac{15}{8}$	$\nearrow \cap$

$f'\left(\frac{1}{100}\right) = -\frac{10^4}{2} + 10^3 < 0$

$f''\left(\frac{1}{100}\right) = 10^6 - \frac{3 \cdot 10^5}{2} = 10^5\left(10 - \frac{3}{2}\right) > 0$

$f'(100) = -\frac{1}{2 \cdot 10^4} + \frac{1}{10^3} = \frac{1}{10^3}\left(1 - \frac{1}{20}\right) > 0$

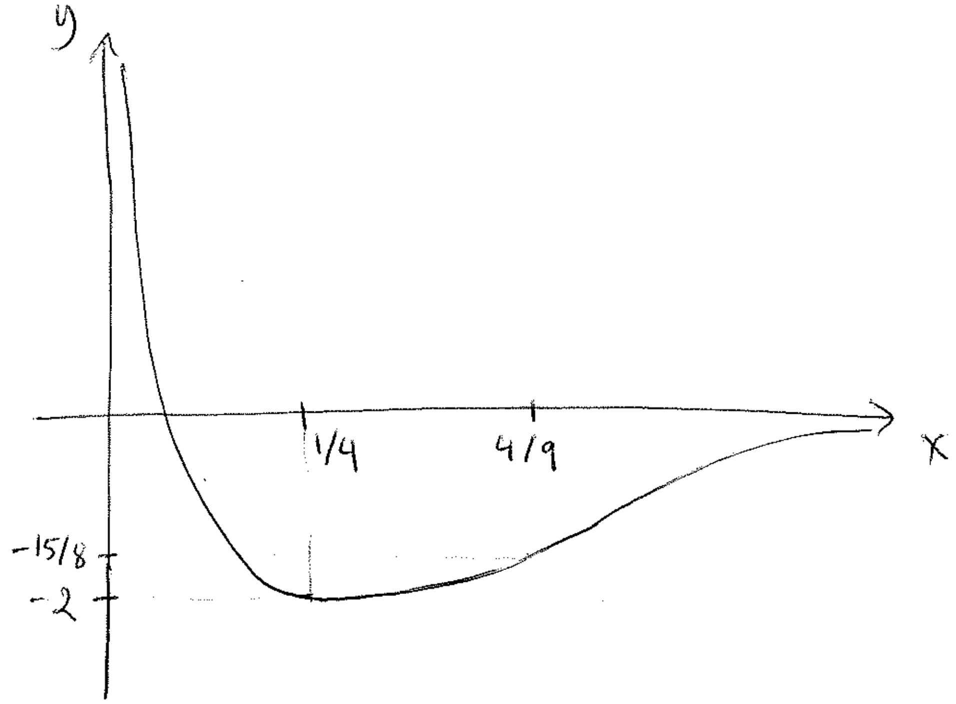
$f''(100) = \frac{1}{10^6} - \frac{3}{2 \cdot 10^5} = \frac{1}{10^5}\left(\frac{1}{10} - \frac{3}{2}\right) < 0$

Steg 5: $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1 - 4\sqrt{x}}{2x} = \infty$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1-4\sqrt{x}}{2x} = \lim_{x \to \infty} \frac{\sqrt{x}\left(\frac{1}{\sqrt{x}} - 4\right)}{2x} =$$

$$= \lim_{x \to \infty} \frac{\frac{1}{\sqrt{x}} - 4}{2\sqrt{x}} = 0$$

Steg 6:



4. Två fall:

<u>$x < -2$</u>: Vi har att:

$$f(x) = 3 + \sin\left(\frac{3}{x+2}\right)$$

Då värdemängden till sin ä $[-1,1]$ medför detta att f kommer att röra sig mellan 2 och 4 då $x<-2$.

<u>$x > -2$</u>: Vi har att:

$$f(x) = \frac{x^2-3}{x+2}$$

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{x^2-3}{x+2} = \infty$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^{\cancel{2}}\left(1-\frac{3}{x^2}\right)}{\cancel{x}\left(1+\frac{2}{x}\right)} = \infty$$

$\Rightarrow$ f har inget största värde.

Återstår att kolla: Finns det något $\tilde{x} \in (-2,\infty)$ s.a. $f(\tilde{x}) < 2$?

$$f'(x) = \frac{2x(x+2)-(x^2-3)}{(x+2)^2} = \frac{\cancel{2}x^2+4x-\cancel{x^2}+3}{(x+2)^2} = \frac{x^2+4x+3}{(x^2+2)^2}$$

$f'(x) = 0 \Rightarrow x^2 + 4x + 3 = 0 \Rightarrow x = -2 \pm \sqrt{4-3} = -2 \pm 1$

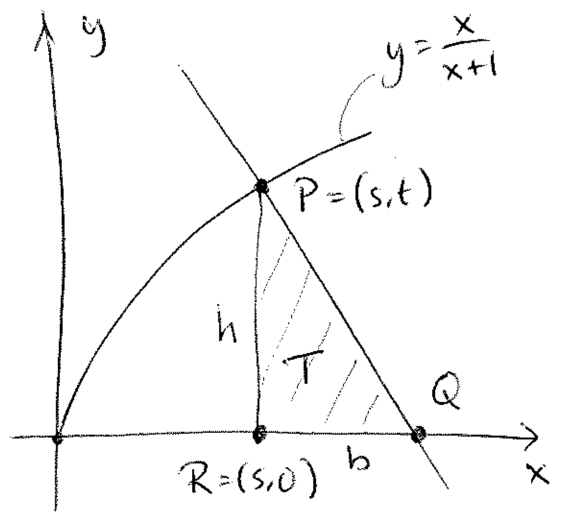
$\Rightarrow x_1 = -1 \quad (x_2 = -3) \notin (-2, \infty)$ Falsk rot!

$x = -1$ min.pkt. då $f(x) \xrightarrow[\substack{x \to -2^+ \\ x \to \infty}]{} \infty$.

$f(-1) = \dfrac{(-1)^2 - 3}{-1+2} = \dfrac{-2}{1} = -2$ globalt min.

$\therefore \quad V_f = [-2, \infty)$.

5.



$$y'(x) = \frac{1}{(x+1)^2} \Rightarrow$$

$$\Rightarrow k_T = \frac{1}{(s+1)^2}$$

$$k_T \cdot k_N = -1 \Rightarrow$$

$$\Rightarrow k_N = -(s+1)^2$$

Normallinje: $y = -(s+1)^2(x-s) + t$

$t = \frac{s}{s+1}$: $y = -(s+1)^2(x-s) + \frac{s}{s+1}$

$$y = 0 \Leftrightarrow (s+1)^2(x-s) = \frac{s}{s+1} \Leftrightarrow x - s = \frac{s}{(s+1)^3}$$

$$\Leftrightarrow x = \frac{s}{(s+1)^3} + s$$

$$\therefore Q = \left(\frac{s}{(s+1)^3} + s, 0 \right)$$

$$b = \frac{s}{(s+1)^3} + \cancel{s} - \cancel{s} = \frac{s}{(s+1)^3}$$

$$h = t = \frac{s}{s+1}$$

$$\Rightarrow \text{Area}(T) = \frac{b \cdot h}{2} = \frac{1}{2} \frac{s}{(s+1)^3} \cdot \frac{s}{s+1} = \frac{s^2}{2(s+1)^4} = f(s)$$

Ser att: $f(s) \xrightarrow[s \to 0^+]{} 0$, $f(s) \xrightarrow[s \to \infty]{} 0$, $f(s) > 0$

$\Rightarrow f(s)$ har en max.pkt. på $(0, \infty)$

$$f'(s) = \frac{4s(s+1)^4 - 8s^2(s+1)^3}{4(s+1)^8} =$$

$$= \frac{\cancel{4}s\cancel{(s+1)^3}(s+1-2s)}{\cancel{4}(s+1)^{\cancel{8}5}} = \frac{s(1-s)}{(s+1)^5}$$

$$f'(s) = 0 \Rightarrow (s_1 = 0) \qquad \underline{\underline{s_2 = 1}}$$

$$P = \left(s, \frac{s}{s+1}\right) \Rightarrow \underline{\underline{P_{max} = \left(1, \frac{1}{2}\right)}}$$

6 (a) Vi söker ett polynom $P^f_{[a,b]}(x) = A + Bx$, där $A, B \in \mathbb{R}$ uppfyller/löser ekvationssystemet

$$\begin{cases} A + Ba = f(a) \\ A + Bb = f(b). \end{cases}$$

Detta ekvationssystem har lösningen:

$$B = \frac{f(b) - f(a)}{b - a}$$

$$A = f(a) - \frac{f(b) - f(a)}{b-a} a.$$

Polynomet blir alltså:

$$P^f_{[a,b]}(x) = f(a) - \frac{f(b) - f(a)}{b - a}(x - a) =$$

$$= f(a) \frac{b - x}{b - a} + f(b) \frac{x - a}{b - a}$$

(b)
```
function y=interpolation(f,x,a,b)
  c=(a+b)/2;
  if x<c
      y=f(a)*(c-x)/(c-a)+f(c)*(x-a)/(c-a);
  else
      y=f(c)*(b-x)/(b-c)+f(b)*(x-c)/(b-c);
  end
```

8. (a) Vet att:

(i) $$\frac{d}{dx}\tan(x) = \frac{1}{\cos^2(x)} = 1 + \tan^2(x)$$

(ii) $$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

(i) + (ii) ger nu att:

$$\frac{d}{dx}\arctan(x) = \frac{1}{1 + (\tan(\arctan x))^2} = \frac{1}{1 + x^2} \quad \blacksquare$$

(b) $$f'(x) = \frac{1}{1+x^2} + \frac{1}{1 + \left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

$\Rightarrow f(x) =$ konstant, men $D_f = \mathbb{R}\setminus\{0\}$ så

$$f(x) = \begin{cases} C_1 & \text{om } x < 0 \\ C_2 & \text{om } x > 0 \end{cases}$$

där C_1 inte behöver vara lika med C_2.

Låt oss kolla om så är fallet:

$$f(-1) = 2\arctan(-1) = 2\cdot\left(-\frac{\pi}{4}\right) = -\frac{\pi}{2}$$

$$f(1) = 2\arctan(1) = 2\cdot\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$$

$$\therefore\ f(x) = \begin{cases} -\frac{\pi}{2} & \text{om } x < 0 \\ \frac{\pi}{2} & \text{om } x > 0 \end{cases}$$

så f antar bara två värden. $\blacksquare$