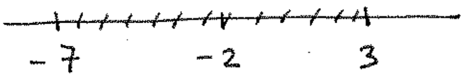
Anonym kod	TMV122/125/177 Inledande Matematik 2016-12-21	Poäng

1. Till nedanstående uppgifter skall korta lösningar redovisas, samt svar anges, på anvisad plats (endast lösningar och svar på detta blad, och på anvisad plats, beaktas).

(a) Bestäm alla reella tal x sådana att $|5x+2| < 5$. (2 p)

Lösning: $|5x+2|<5 \Leftrightarrow |5x-(-2)|<5$



$\Rightarrow -7 < 5x < 3 \Leftrightarrow$

$\Leftrightarrow -\frac{7}{5} < x < \frac{3}{5}$

Svar: $x \in \left(-\frac{7}{5}, \frac{3}{5}\right)$

(b) Om $f'(4) = 3$ beräkna, (2 p)

$$\left(\frac{d}{dx} f(\sqrt{36-5x^2})\right)\Big|_{x=2}$$

Lösning: $\left(\frac{d}{dx} f(\sqrt{36-5x^2})\right)\Big|_{x=2} = \left(f'(\sqrt{36-5x^2}) \cdot \frac{1}{2} \cdot \frac{-10x}{\sqrt{36-5x^2}}\right)\Big|_{x=2}$

$= f'(\sqrt{36-20}) \cdot \frac{-5 \cdot 2}{\sqrt{36-20}} = f'(4) \cdot \frac{-5 \cdot 2}{4} = \frac{3 \cdot (-5)}{2}$

Svar: $-15/2$

(c) Bestäm normallinjen till kurvan, (2 p)

$$y\cos(x) = 1 + \sin(xy)$$

i punkten $(0, 1)$.

Lösning: Derivera implicit m.a.p. x:

$y'\cos(x) - y\sin(x) = \cos(xy)(y + xy')$

$\underline{(0,1)}$: $y' = 1 \cdot (1 + 0 \cdot y') \Leftrightarrow y'_T = 1 \Rightarrow y'_N = -1$

$\Rightarrow y = -1 \cdot (x - 0) + 1 = -x + 1$

Svar: $y = -x + 1$

(d) Bestäm värdemängden till funktionen, (2 p)

$$f(x) = e^{\sin(x)\cos(x)}.$$

Lösning: $f(x) = e^{\sin(x)\cos(x)} = e^{\frac{1}{2}\sin(2x)}$

$\sin(2x) \in [-1, 1] \implies V_f = [e^{-\frac{1}{2}}, e^{\frac{1}{2}}]$

Svar: $V_f = [e^{-\frac{1}{2}}, e^{\frac{1}{2}}]$

(e) Beräkna gränsvärdena (1+1 p)

(i) $\lim_{x\to 0} \dfrac{e^{x^2}-1}{\cos(x)-1}$

(ii) $\lim_{x\to\infty} \left(1-\dfrac{2}{x}\right)^{3x}$

Lösning: (i) $\lim_{x\to 0} \dfrac{e^{x^2}-1}{\cos(x)-1} \overset{\left[\frac{0}{0}\right]}{=} \{\ell'\text{Hôpital}\} = \lim_{x\to 0} \dfrac{2xe^{x^2}}{-\sin(x)} =$

$= \lim_{x\to 0} -2\dfrac{1}{\frac{\sin(x)}{x}} e^{x^2} = -2$

(ii) Låt $y = \left(1-\frac{2}{x}\right)^{3x}$. Då gäller att:

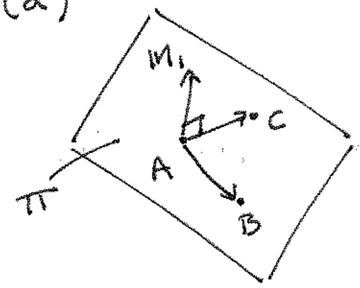
$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} 3x\ln\left(1-\frac{2}{x}\right) \overset{[\infty\cdot 0]}{=} \lim_{x\to\infty} 3\cdot\dfrac{\ln\left(1-\frac{2}{x}\right)}{\frac{1}{x}} \overset{\left[\frac{0}{0}\right]}{=}$

$= \{\ell'\text{Hôpital}\} = \lim_{x\to\infty} 3\cdot\dfrac{\frac{1}{1-\frac{2}{x}}\cdot\left(\frac{2}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} -6\dfrac{1}{1-\frac{2}{x}} = -6$

$\therefore \ln(y) \to -6$ då $x\to\infty \Rightarrow y \to e^{-6}$ då $x\to\infty$ då e^x är kont. i $x=-6$

Svar: (i) -2 (ii) e^{-6}

2. (a)



Låt $A=(1,1,0)$, $B=(1,0,1)$ och $C=(0,1,1)$

Ser att $n_1 \parallel (\vec{AB}\times\vec{AC}) \Rightarrow$

$\Rightarrow n_1 = \vec{AB}\times\vec{AC}$ (längden av n_1 är irrelevant!)

$\vec{AB} = B-A = (0,-1,1)$, $\vec{AC} = (-1,0,1)$

$$\Rightarrow n_1 = \vec{AB}\times\vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = (-1,-1,-1)$$

$\Rightarrow \pi: -x-y-z = D$

$\underline{A\in\pi}$: $D = -1-1-0 = -2$

$\therefore \pi: -x-y-z=-2 \iff x+y+z=2$

xy-planet har normalen $n_2 = (0,0,1)$

Vinkeln mellan π och xy-planet =

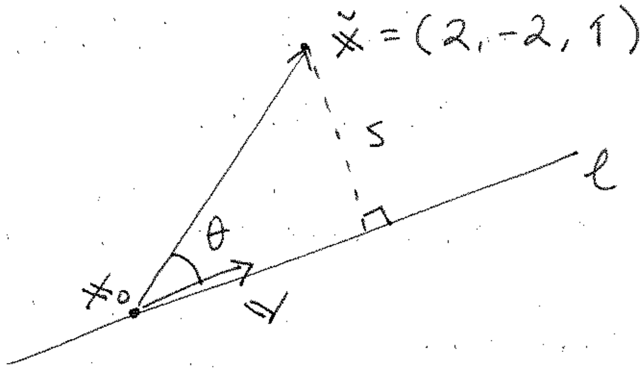
= vinkeln mellan n_1 och n_2

Vet att: $n_1\cdot n_2 = |n_1||n_2|\cos\theta$

$$\Rightarrow \theta = \arccos\left(\frac{n_1\cdot n_2}{|n_1||n_2|}\right) = \underline{\underline{\arccos\left(\frac{-1}{\sqrt{3}}\right)}}$$ ($\arccos\left(\frac{1}{\sqrt{3}}\right)$ också ok!)

(b) $\ell: \begin{cases} y=1 \\ x-z=1 \end{cases}$ Om $z=t$ så $\ell: \begin{cases} x=1+t \\ y=1 \\ z=t \end{cases}$, $t\in\mathbb{R}$

$$\iff \ell: \mathbf{x} = \underbrace{(1,1,0)}_{\mathbf{x}_0} + t\underbrace{(1,0,1)}_{\mathbf{v}},\ t\in\mathbb{R}$$



$$\tilde{x} - x_0 = (1,-3,1)$$

$$s = |\tilde{x} - x_0|\sin\theta = \frac{|v||\tilde{x} - x_0|\sin\theta}{|v|} = \frac{|v \times (\tilde{x} - x_0)|}{|v|}$$

$$v \times (\tilde{x} - x_0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 1 & -3 & 1 \end{vmatrix} = (3, 0, -3)$$

$$\Rightarrow \quad s = \frac{|(3,0,-3)|}{|(1,0,1)|} = \frac{\sqrt{18}}{\sqrt{2}} = \frac{\sqrt{2}\sqrt{9}}{\sqrt{2}} = \underline{\underline{3}}$$

3. $f(x) = \dfrac{x^3}{x^2+1}$

<u>Steg 1</u>: $D_f = \mathbb{R}$

<u>Steg 2</u>: $f'(x) = \dfrac{3x^2(x^2+1) - x^3 \cdot 2x}{(x^2+1)^2} = \dfrac{x^2(x^2+3)}{(x^2+1)^2}$

$f'(x) = 0 \Rightarrow x = 0$ kritisk punkt

Ser att $f'(x) \geq 0 \ \forall x \in D_f \Rightarrow f$ alltid växande

<u>Steg 3</u>: $f''(x) = \dfrac{(4x^3+6x)(x^2+1)^2 - (x^4+3x^2)2(x^2+1)2x}{(x^2+1)^4} =$

$= \dfrac{2x\cancel{(x^2+1)}\left((2x^2+3)(x^2+1) - 2x^4 - 6x^2\right)}{(x^2+1)^{\cancel{4}3}} =$

$= \dfrac{2x\left(\cancel{2x^4} + 5x^2 + 3 - \cancel{2x^4} - 6x^2\right)}{(x^2+1)^3} = \dfrac{2x(3-x^2)}{(x^2+1)^3}$

$f''(x) = 0 \Rightarrow x_1 = 0,\ x_2 = -\sqrt{3},\ x_3 = \sqrt{3}$ ev. infl. pkter.

<u>Steg 4</u>:

		$-\sqrt{3}$		0		$\sqrt{3}$	
f'	$+++$		$+++$	0	$+++$		$+++$
f''	$+++$	0	$---$	0	$+++$	0	$---$
f	$\nearrow \cup$	infl. pkt.	$\nearrow \cap$	infl. pkt.	$\nearrow \cup$	infl. pkt.	$\nearrow \cap$

$f(-\sqrt{3}) = -\dfrac{3\sqrt{3}}{4}$, $f(0) = 0$, $f(\sqrt{3}) = \dfrac{3\sqrt{3}}{4}$

<u>Steg 5</u>: I. <u>Lodräta asymptoter</u>: $D_f = \mathbb{R} \Rightarrow$

$\Rightarrow$ Inga lodräta asymptoter!

II. Vågräta asymptoter:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{1+\frac{1}{x^2}} = \infty$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{1+\frac{1}{x^2}} = -\infty$$

$\Rightarrow$ Inga vågräta asymptoter!

III. Sneda asymptoter:

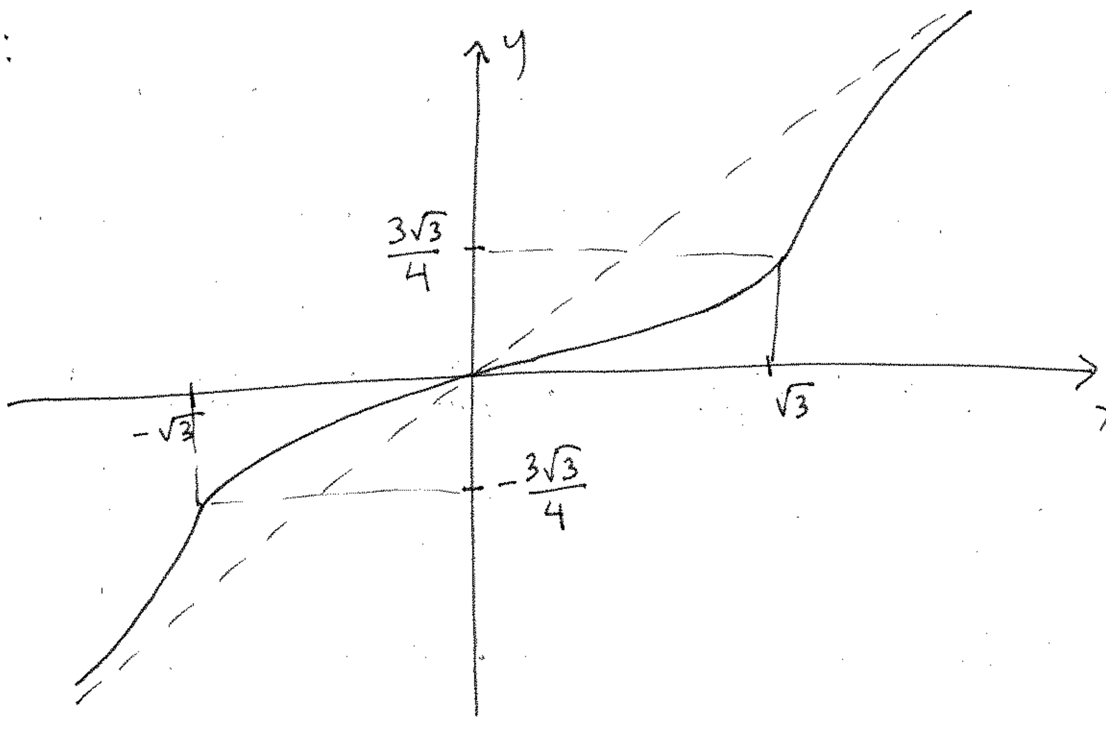
$$k_1 = k_2 = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{1}{1+\frac{1}{x^2}} = 1$$

$$m_1 = m_2 = \lim_{x\to\pm\infty} \left(f(x) - 1\cdot x\right) =$$

$$= \lim_{x\to\pm\infty} \left(\frac{x^3}{1+x^2} - x\cdot\frac{1+x^2}{1+x^2}\right) = \lim_{x\to\pm\infty} -\frac{x}{1+x^2} = 0$$

$\Rightarrow$ $y = x$ sned asymptot!

Steg 6:



4. $f(x) = \frac{1}{x} + 2\ln(x+1)$

$D_f = (-1, 0) \cup (0, \infty)$

Ser att:

$f(x) \to -\infty$ då $x \to -1^+$ eller $x \to 0^-$ $(*)$

$f(x) \to \infty$ då $x \to 0^+$ eller $x \to \infty$ $(**)$

(Obs! Detta innebär <u>inte</u> att $V_f = \mathbb{R}$!)

$f'(x) = -\frac{1}{x^2} + \frac{2}{x+1}$

$f'(x) = 0 \iff \frac{2}{x+1} = \frac{1}{x^2} \iff 2x^2 = x + 1 \iff$

$\iff x^2 - \frac{1}{2}x - \frac{1}{2} = 0$

$\Rightarrow x = \frac{1}{4} \pm \sqrt{\frac{1}{16} + \frac{1}{2}} = \frac{1}{4} \pm \frac{3}{4}$

$\Rightarrow x_1 = -\frac{1}{2}, \quad x_2 = 1$

$(*) \Rightarrow x_1 = -\frac{1}{2}$ lokalt max.

$(**) \Rightarrow x_2 = 1$ lokalt min.

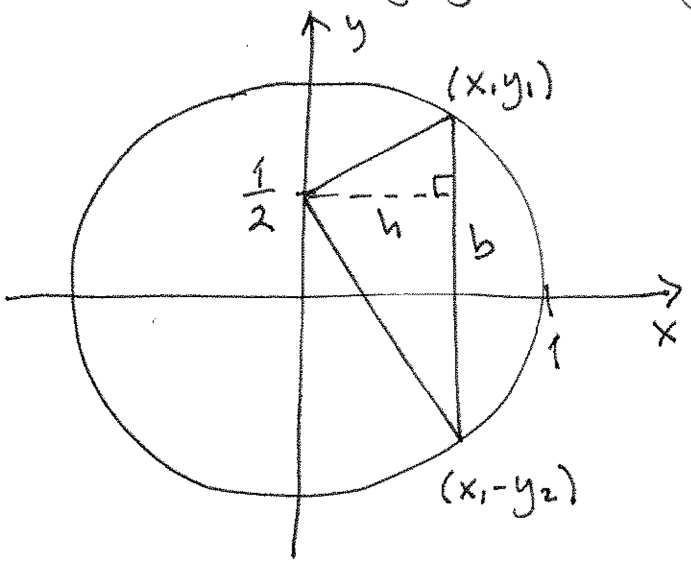
$f(-\frac{1}{2}) = -2 + 2\ln(\frac{1}{2}) = -2 - 2\ln(2) < 0$

$f(1) = 1 + 2\ln(2) > 0$

$\Rightarrow f(-\frac{1}{2}) < f(1)$

$\therefore V_f = (-\infty, -2-2\ln(2)] \cup [1 + 2\ln(2), \infty)$

5. Vi har följande figur



$$\text{Area} = \frac{b \cdot h}{2}$$

Vi ser att:

$h = x$

$b = y_1 + y_2$

Då (x, y_1) och $(x, -y_2)$ ligger på enhetscirkeln har vi att

$$y_1 = y_2 = \sqrt{1-x^2}$$

$$\therefore \quad \text{Area} = A(x) = \frac{x \cdot \cancel{2}\sqrt{1-x^2}}{\cancel{2}} = x\sqrt{1-x^2}, \quad 0 \leq x \leq 1$$

$$A'(x) = \sqrt{1-x^2} + x \cdot \frac{1}{2} \cdot \frac{-2x}{\sqrt{1-x^2}} =$$

$$= \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}$$

$$A'(x) = 0 \implies 1 - 2x^2 = 0 \iff x^2 = \frac{1}{2}$$

$$\implies x = \underset{(-)}{+} \frac{1}{\sqrt{2}}$$

$$A(0) = A(1) = 0 \implies x = \frac{1}{\sqrt{2}} \text{ max. pkt.}$$

$$A\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \cdot \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \underline{\underline{\frac{1}{2} \text{ a.e.}}}$$

6. Låt $f(x) = x^{1/x}$ och $g(x) = a$

Antal lösningar till ekvationen $x^{1/x} = a$

$\Longleftrightarrow$

Antal skärningspunkter mellan f och g för olika värden på a.

Rita grafen till f!

Steg 1: $f(x) = x^{1/x} = e^{\frac{\ln(x)}{x}}$, $D_f = (0, \infty)$

Steg 2: $$f'(x) = e^{\frac{\ln(x)}{x}} \left(\frac{\frac{1}{x} \cdot x - \ln(x)}{x^2} \right) = e^{\frac{\ln(x)}{x}} \cdot \frac{1 - \ln(x)}{x^2}$$

$$f'(x) = 0 \Rightarrow 1 - \ln(x) = 0 \Leftrightarrow x = e$$

Steg 3: f'' behövs ej!

Steg 4:

		e	
f'	$+++$	0	$---$
f	$\nearrow$		$\searrow$

$\Rightarrow x = e$ max.pkt., $f(e) = e^{1/e}$

Steg 5: $$\lim_{x \to 0^+} \frac{\ln(x)}{x} = [\infty \cdot (-\infty)] = -\infty$$

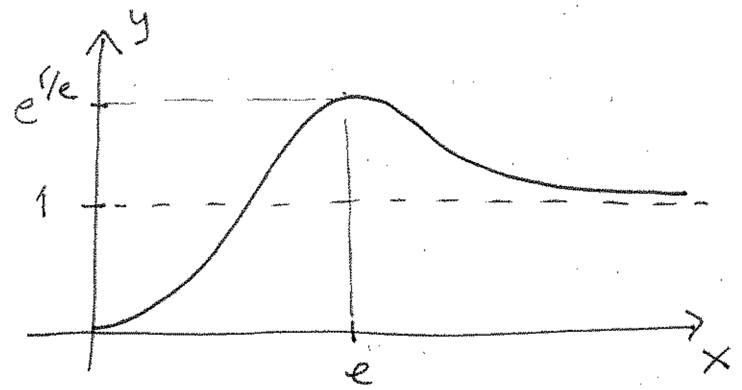
$$\Rightarrow \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{\frac{\ln(x)}{x}} = 0$$

$$\lim_{x \to \infty} \frac{\ln(x)}{x} \leftarrow \left[\frac{\infty}{\infty}\right] = \{\text{l'Hôpital}\} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$$

$$\Rightarrow \lim_{x \to \infty} f(x) = \lim_{x \to \infty} e^{\frac{\ln(x)}{x}} = e^0 = 1 \quad \text{då } e^x \text{ kont. i } x = 0$$

Steg 6: Har att $e > 2$ och $\frac{1}{e} > \frac{1}{3}$

$\Rightarrow e^{1/e} > 2^{1/e} > 2^{1/3} > 1$ så $e^{1/e} > 1$



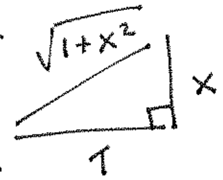
Av figuren framgår att vi har:

1 lösning då $a \in (0,1] \cup \{e^{1/e}\}$

2 lösningar då $a \in (1, e^{1/e})$

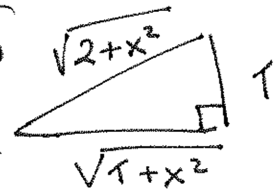
Inga lösningar då $a \in \mathbb{R} \setminus (0, e^{1/e}]$

8. $\sin\left(\frac{\pi}{2}-\arctan x\right)=\underbrace{\sin\frac{\pi}{2}}_{=1}\cos(\arctan x)-\underbrace{\cos\frac{\pi}{2}}_{=0}\sin(\arctan x)=$

$=\cos(\arctan x)=\Big\{$ 

$\Big\}=\dfrac{1}{\sqrt{1+x^2}}$

$\Rightarrow \cos\left(\arctan\left(\sin\left(\frac{\pi}{2}-\arctan(x)\right)\right)\right)=$

$=\cos\left(\arctan\left(\frac{1}{\sqrt{1+x^2}}\right)\right)=\Big\{$ 

$\Big\}=$

$$=\frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}.$$

$\blacksquare$