1a) (i) $\lim_{x\to\infty} \left(\sqrt{4x^2+3x} - 2x\right) = \lim_{x\to\infty} \frac{\left(\sqrt{4x^2+3x} - 2x\right)\left(\sqrt{4x^2+3x} + 2x\right)}{\sqrt{4x^2+3x} + 2x}$

$= \lim_{x\to\infty} \frac{\cancel{4x^2} + 3x - \cancel{4x^2}}{\sqrt{4x^2+3x} + 2x} = \left\{ |x| = x \text{ då } x \to \infty \right\}$

$= \lim_{x\to\infty} \frac{3\cancel{x}}{\cancel{x}\sqrt{4 + \underbrace{\tfrac{3}{x}}_{\to 0}} + 2\cancel{x}} = \frac{3}{2+2} = \frac{3}{4}$

(ii) $\lim_{x\to 0} \frac{\sin(\pi x)}{x^2 - x} = \lim_{x\to 0} \pi \underbrace{\frac{\sin \pi x}{\pi x}}_{\xrightarrow[x\to 0]{} 1} \cdot \underbrace{\frac{1}{(x-1)}}_{\xrightarrow[x\to 0]{} -1} = -\pi$

Svar: (i) $\frac{3}{4}$ (ii) $-\pi$

b) $x^2y + xy^3 = 2$ Derivera implicit m.a.p. x

$\overset{\frac{d}{dx}}{\Longrightarrow} \; 2xy + x^2y' + y^3 + 3xy^2y' = 0$

$\Rightarrow \; y'(x^2 + 3xy^2) + (2xy + y^3) = 0$

$\Rightarrow \; y' = -\frac{(2xy + y^3)}{(x^2 + 3xy^2)}$

I punkten $(x,y) = (1,1)$ har vi $y' = -\frac{2+1}{1+3} = -\frac{3}{4}$

$\Rightarrow$ Tangentens ekv. $y = -\frac{3}{4}x + m$

Insättning av $(x,y) = (1,1)$ ger $m = 1 + \frac{3}{4} = \frac{7}{4}$

$\Rightarrow \; y = -\frac{3}{4}x + \frac{7}{4}$

Svar: $y = -\frac{3}{4}x + \frac{7}{4}$

c) $\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 1 & a(a-1) & a \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & a(a-1)-2 & a-2 \end{array}\right]$

$a(a-1)-2 = a^2-a-2 = (a-2)(a+1)$

$\Rightarrow$ Entydig løsning då $(a-2)(a+1) \neq 0$

$\Leftrightarrow \quad a \neq -1, 2$

Svar: Entydig løsning for $a \neq -1, 2$

d) $$\cos\phi = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{1\cdot 0 + 1\cdot 1 + 2\cdot 1}{\sqrt{1^2+1^2+2^2}\sqrt{1^2+1^2}} = \frac{3}{\sqrt{6}\sqrt{2}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \phi = \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

Svar: $\phi = \frac{\pi}{6}$

e) $$f'(x) = \frac{1}{\sin x \cdot \cos x}\left(\cos^2 x - \sin^2 x\right)$$

$$f'\left(\frac{\pi}{3}\right) = \frac{1}{\frac{\sqrt{3}}{2}\cdot\frac{1}{2}}\left(\left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2\right) = \frac{4}{\sqrt{3}}\cdot\left(-\frac{1}{2}\right) = -\frac{2}{\sqrt{3}}$$

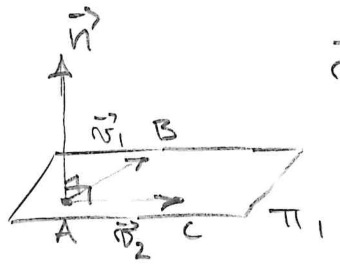
Svar: $f'\left(\frac{\pi}{3}\right) = -\frac{2}{\sqrt{3}}$

f) Vektorprojeksjonen ges av

$$\vec{u}_{\vec{v}} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} = \frac{3}{2}\vec{v} = \left(0, \frac{3}{2}, \frac{3}{2}\right)$$

Svar: $\vec{u}_{\vec{v}} = \left(0, \frac{3}{2}, \frac{3}{2}\right)$

2 a) Två vektorer i planet π_1 ges av



$$\vec{v}_1 = \vec{AB} = (1,0,0) - (2,1,-1) = (-1,-1,1)$$
$$\vec{v}_2 = \vec{AC} = (3,-1,2) - (2,1,-1) = (1,-2,3)$$

Normalen $\vec{n}$ uppfyller $\vec{n} \perp \vec{v}_1$, $\vec{n} \perp \vec{v}_2$

$$\Rightarrow \vec{n} \parallel \vec{v}_1 \times \vec{v}_2$$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -1 & -1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = (-3+2, 3+1, 2+1) = (-1,4,3)$$

Längden $|\vec{n}|$ irrelevant, välj $\vec{n} = \vec{v}_1 \times \vec{v}_2 = (-1,4,3)$

$$\Rightarrow \pi_1 : -x + 4y + 3z = D_{\pi_1}$$

För att bestämma D_{π_1} behöver vi punkt i π_1,

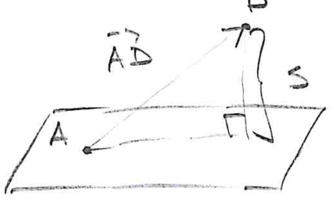
välj tex. $\vec{B} = (1,0,0)$

$$\Rightarrow D_{\pi_1} = \vec{n} \cdot \vec{B} = -1 \cdot 1 + 4 \cdot 0 + 3 \cdot 0 = -1$$

$$\Rightarrow \pi_1 : -x + 4y + 3z = -1$$

Svar: $\pi_1 : -x + 4y + 3z = -1$

b)



Välj någon punkt i planet, tex $A = (2,1,-1)$

Minsta avståndet ges av

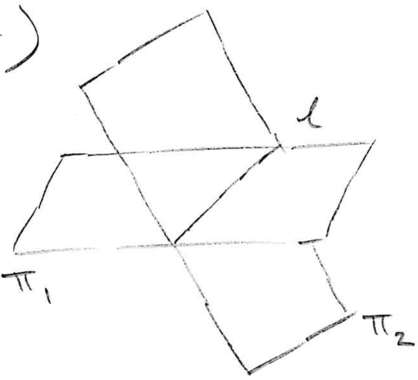
$$s = \frac{|\vec{AD} \cdot \vec{n}|}{|\vec{n}|}$$

$$\vec{AD} = (0,2,2) - (2,1,-1) = (-2,1,3)$$

$$\Rightarrow \quad s = \frac{|(-2,1,3)\cdot(-1,4,3)|}{\sqrt{1+16+9}} = \frac{|2+4+9|}{\sqrt{26}} = \frac{15}{\sqrt{26}}$$

Svar: Avståndet är $s = \dfrac{15}{\sqrt{26}}$ l.e.

c)



Skärningslinjen ℓ ges av lösningen till systemet

$$\begin{cases} -x+4y+3z = -1 \\ 2x - y + z = 0 \end{cases}$$

π_2: $2x - y + z = 0$

$$\Rightarrow \left[\begin{array}{ccc|c} -1 & 4 & 3 & -1 \\ 2 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 4 & 3 & -1 \\ 0 & 7 & 7 & -2 \end{array}\right] \begin{array}{l} \cdot -1 \\ \cdot \frac{1}{7} \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -4 & -3 & 1 \\ 0 & 1 & 1 & -\frac{2}{7} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -\frac{1}{7} \\ 0 & 1 & 1 & -\frac{2}{7} \end{array}\right]$$

t fri variabel

$$\Rightarrow \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad t \in \mathbb{R}$$

Svar: Skärningslinjen ges av

$$\ell: \quad \vec{x} = \frac{1}{7}\begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad t \in \mathbb{R}$$

3) Rita grafen till $f(x) = \dfrac{e^x}{x-2}$

Steg 1: $D_f = \mathbb{R} \setminus \{2\}$

Steg 2: $f'(x) = \dfrac{e^x(x-2) - e^x}{(x-2)^2} = \dfrac{e^x(x-3)}{(x-2)^2}$

$f'(x) = 0 \Rightarrow x = 3$ kritisk punkt

Steg 3: $f''(x) = \dfrac{(e^x(x-3) + e^x)(x-2)^2 - 2(x-2)e^x(x-3)}{(x-2)^4}$

$= \dfrac{e^x(x-2)^{\cancel{3}2} - 2e^x\cancel{(x-2)}(x-3)}{(x-2)^{\cancel{4}3}} = \dfrac{e^x(x^2-4x+4-2x+6)}{(x-2)^3}$

$= \dfrac{e^x(x^2-6x+10)}{(x-2)^3} \neq 0$ ty $x^2-6x+10 > 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow$ Inga inflektionspunkter.

Steg 4: Teckentabell

x		2		3	
$x-2$	$-$		$+$		$+$
$x-3$	$-$		$-$		$+$
f'	$-$	ej def.	$-$	0	$+$
f''	$-$	"	$+$		$+$
f	$\searrow$ $\cap$	"	$\searrow$ $\cup$	e^3 lokalt min	$\nearrow$ $\cup$

$f(3) = \dfrac{e^3}{3-2} = e^3$

Steg 5: Asymptoter

Lodräta:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{e^x}{\underbrace{x-2}_{<0}} = -\infty$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{e^x}{\underbrace{x-2}_{>0}} = +\infty$$

$\Rightarrow$ $x = 2$ lodrät asymptot

Vågräta:

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{e^x}{x-2} = 0$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{e^x}{x-2} = \infty$$

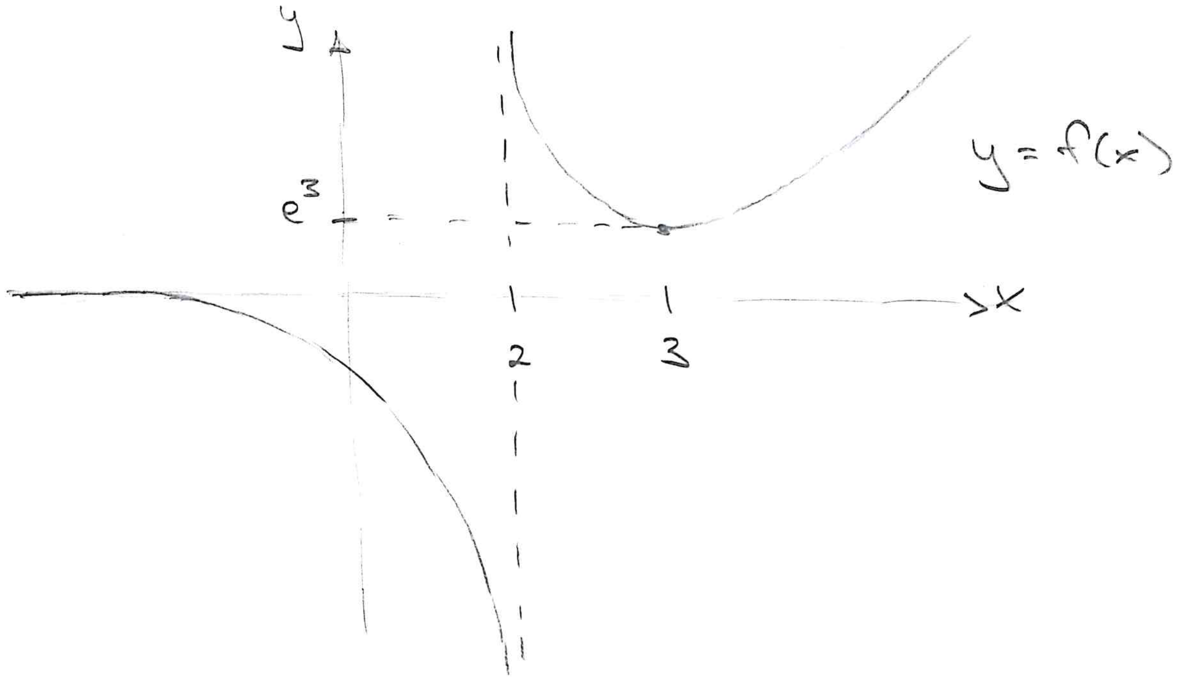
$\Rightarrow$ $y = 0$ vågrät asymptot då $x \to -\infty$

Sneda:

$$k = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{e^x}{x(x-2)} = \infty$$

$\Rightarrow$ Ingen sned asymptot då $x \to \infty$

Steg 6: Skiss av $f(x)$



4) $f(x) = \frac{1}{x+1} + \ln(x+3)$

$D_f = (-3, \infty) \setminus \{-1\} = (-3,-1) \cup (-1, \infty)$

För att bestämme värdemängden V_f studerar vi

$$f'(x) = -\frac{1}{(x+1)^2} + \frac{1}{x+3} = \frac{x^2+2x+1-(x+3)}{(x+1)^2(x+3)}$$

$$= \frac{x^2+x-2}{(x+1)^2(x+3)} = \frac{(x+2)(x-1)}{(x+1)^2(x+3)}$$

$f'(x) = 0 \Rightarrow x = -2, 1$ kritiska punkter

Teckentabell för f, f':

x	-3		-2		-1		1	
$x+3$	0	+		+		+		+
$x+2$		−	0	+		+		+
$x+1$		−		−	0	+		+
$x-1$		−		−		−	0	+
f'	ej def	+	0	−	ej def	−	0	+
f	ej def	↗	lokalt max	↘	ej def	↘	lokalt min	↗

Vidare har vi

$\lim_{x\to -3^+} f(x) = -\infty$, $\lim_{x\to -1^-} f(x) = -\infty$, $\lim_{x\to -1^+} f(x) = \infty$

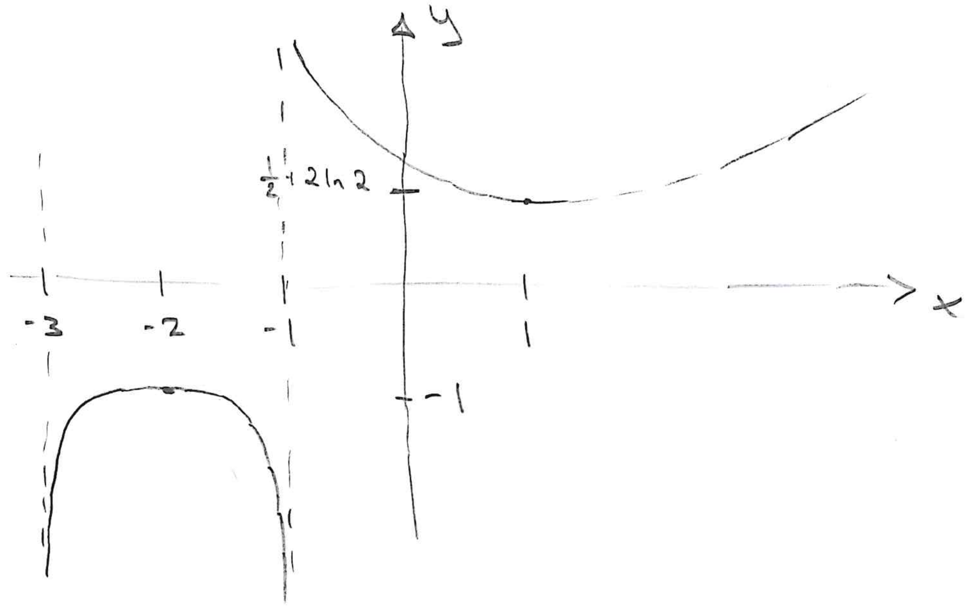
samt $\lim_{x\to\infty} f(x) = \infty$

Lokala extremvärden ges av

$$f(-2) = \frac{1}{-2+1} + \ln(-2+3) = -1 + \underbrace{\ln 1}_{=0} = -1$$

$$f(1) = \frac{1}{1+1} + \ln(1+3) = \frac{1}{2} + \ln 4 = \frac{1}{2} + 2\ln 2$$

Vi kan skissa funktionens graf.



$$\Rightarrow \quad V_f = (-\infty, -1] \cup [\tfrac{1}{2} + 2\ln 2, \infty)$$

<u>Svar:</u> $D_f = (-3, -1) \cup (-1, \infty)$

$V_f = (-\infty, -1] \cup [\tfrac{1}{2} + 2\ln 2, \infty)$

5 a) $f(x) = \tanh(x) = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$ $\quad D_f = \mathbb{R}$

$$f'(x) = \frac{(e^{x}+e^{-x})(e^{x}+e^{-x}) - (e^{x}-e^{-x})(e^{x}-e^{-x})}{(e^{x}+e^{-x})^2}$$

$$= \frac{\cancel{e^{2x}} + \cancel{e^{-2x}} + 2 - (\cancel{e^{2x}} + \cancel{e^{-2x}} - 2)}{(e^{x}+e^{-x})^2}$$

$$= \frac{4}{(e^{x}+e^{-x})^2} > 0 \quad \forall x \in \mathbb{R}$$

$\Rightarrow$ $f(x)$ strängt växande på $\mathbb{R}$

$\Rightarrow$ $f(x)$ injektiv på $\mathbb{R}$

$\Rightarrow$ $f(x)$ inverterbar på $\mathbb{R}$ $\boxed{B}$

b) $\lim\limits_{x \to -\infty} f(x) = -1$, $\lim\limits_{x \to \infty} f(x) = 1$ $\Rightarrow V_f = (-1, 1)$

$\Rightarrow D_{f^{-1}} = (-1, 1)$, $V_{f^{-1}} = \mathbb{R}$

Låt $x = f(y) \Leftrightarrow y = f^{-1}(x)$

$$x = f(y) = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}}$$

$\Rightarrow$ $x e^{y} + x e^{-y} = e^{y} - e^{-y}$

$\Rightarrow$ $e^{2y}(x-1) + (x+1) = 0$

$\Rightarrow$ $e^{2y} = -\dfrac{x+1}{x-1} = \dfrac{1+x}{1-x} > 0$ för $x \in (-1, 1)$

$$\Rightarrow \quad y = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) = f^{-1}(x)$$

<u>Svar:</u> $f^{-1}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \quad , \quad x \in (-1, 1)$

6 a) Se "Theorem 14" i kap 2.8

i "Calculus" av Adams.

b) $V(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - 2x^2 - 4x + 10$

$V'(x) = x^3 + x^2 - 4x - 4 = (x+1)(x^2-4) = (x+1)(x+2)(x-2)$

$V'(x) = 0 \Rightarrow x = -2, -1, 2$ m kritiska punkter

Teckentabell för V, V':

x		-2		-1		2	
V'	$-$	0	$+$	0	$-$	0	$+$
V	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$
		lokalt min		lokalt max		lokalt min	

Eftersom $\lim_{x \to \pm\infty} V(x) = \infty$ antas globalt

minimum i någon av de lokala min.punkterna:

$V(-2) = \frac{16}{4} - \frac{8}{3} - 2 \cdot 4 + 4 \cdot 2 + 10 = 14 - \frac{8}{3} = \frac{34}{3}$ J

$V(2) = \frac{16}{4} + \frac{8}{3} - 2 \cdot 4 - 4 \cdot 2 + 10 = \frac{8}{3} - 2 = \frac{2}{3}$ J

Svar: Positionen som minimerar $V(x)$ är

$x = 2$ m.

7a) Se "Theorem 15" i kap 2.8 i "Calculus" av Adams.

b) Se "Theorem 11" samt bevis på s. 143 i kap 2.8 i "Calculus" av Adams.