Matematisk analys i en variabel E1 (TMV136) 071221

1. a) Partialbråksuppdelning: $\dfrac{1}{x(x-3)} = \dfrac{A}{x} + \dfrac{B}{x-3} = \{\text{HP}\} = \dfrac{-1/3}{x} + \dfrac{1/3}{x-3} \Rightarrow \displaystyle\int \dfrac{1}{x(x-3)}\,dx =$
$= \dfrac{1}{3}(\ln|x-3| - \ln|x|) + C = \dfrac{1}{3}\ln|\dfrac{x-3}{x}| + C.$ b) $\displaystyle\int \dfrac{\cos x}{1+\sin^2 x}\,dx = \left[\begin{array}{c} t = \sin x \\ dt = \cos x dx \end{array}\right] = \int \dfrac{dt}{1+t^2} =$
$= \arctan t + C = \arctan(\sin x) + C.$

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2. $y' + xy = x$ är 1:a ord. linjär (och även separabel: $\frac{1}{1-y}\,dy = x\,dx$) med IF: $e^{\int x\,dx} = e^{x^2/2} \Rightarrow \frac{d}{dx}(e^{x^2/2}y) = xe^{x^2/2} \Rightarrow e^{x^2/2}y = \int \frac{d}{dx}(e^{x^2/2}y)\,dx = \int xe^{x^2/2} = e^{x^2/2} + C \Rightarrow y = 1 + Ce^{-x^2/2} \Rightarrow 7 = y(0) = 1 + C \Rightarrow C = 6 \Rightarrow y = 1 + 6e^{-x^2/2}$.

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3. a) Volymen $= \int_0^2 \pi(x(2-x))^2\,dx = \pi\int_0^2 x^2(4-4x+x^2)\,dx = \pi[\frac{4}{3}x^3 - x^4 + \frac{1}{5}x^5]_0^2 = \ldots = \frac{16\pi}{15}$.
b) Volymen $= \int_0^2 2\pi x f(x)\,dx = \int_0^2 2\pi x^2(2-x)\,dx = 2\pi[\frac{2}{3}x^3 - \frac{1}{4}x^4]_0^2 = \frac{8\pi}{3}$.

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4. $\frac{1}{x(x^2-1)} = \frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \{\text{HP}\} = \frac{-1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1} \Rightarrow \int_2^\infty \frac{dx}{x(x^2-1)}\,dx = \lim_{R\to\infty}\int_2^R \frac{-1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1}\,dx = \lim_{R\to\infty}\Big[-\ln|x| + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1|\Big]_2^R = \lim_{R\to\infty}\Big[\frac{1}{2}\ln\frac{x^2-1}{x^2}\Big]_2^R = \frac{1}{2}(\lim_{R\to\infty}\ln\frac{R^2-1}{R^2} - \ln\frac{3}{4}) = \frac{1}{2}(\ln 1 - \ln\frac{3}{4}) = \frac{1}{2}\ln\frac{4}{3}$

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5. $e^{2x} - 1 = 1 + 2x + \mathcal{O}(x^2) - 1 = 2x + \mathcal{O}(x^2)$, $\ln(1+x^3) = x^3 + \mathcal{O}(x^6)$, $\cos(3x) - 1 = 1 - \frac{(3x)^2}{2} + \mathcal{O}(x^4) - 1 = -\frac{9x^2}{2} + \mathcal{O}(x^4)$. Detta ger $(\cos(3x)-1)^2 = \frac{81}{4}x^4 + \mathcal{O}(x^6)$ så att $\lim_{x\to 0}\frac{(e^{2x}-1)\ln(1+x^3)}{(\cos(3x)-1)^2} = \lim_{x\to 0}\frac{2x^4 + \mathcal{O}(x^5)}{\frac{81}{4}x^4 + \mathcal{O}(x^6)} = \lim_{x\to 0}\frac{2+\mathcal{O}(x)}{\frac{81}{4} + \mathcal{O}(x^2)} = \frac{2+0}{\frac{81}{4}+0} = \frac{8}{81}$.

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6. $y'' + 4y' + 4y = e^{2x}$ har lösn. $y = y_p + y_h$. För y_h : Kar. ekv.: $0 = r^2 - 4r + 4 = (r-2)^2 \Rightarrow r_{1,2} = 2 \Rightarrow y_h = (A + Bx)e^{2x}$. Ansätt $y_p = Cx^2e^{2x}$ ty e^{2x} och xe^{2x} förekommer i y_h. Detta ger $y_p' = (C2x + 2Cx^2)e^{2x}$, $y_p'' = (2C + 8Cx + 4Cx^2)e^{2x}$. Insättning i ekv. ger $e^{2x} = [(2C + 8Cx + 4Cx^2) - 4(2Cx + 2Cx^2) + 4Cx^2]e^{2x} \Rightarrow 2C = 1$. Härav fås att: $y_p = \frac{1}{2}x^2e^{2x}$ så att $y = y_p + y_h = \frac{1}{2}x^2e^{2x} + (A + Bx)e^{2x} = (\frac{1}{2}x^2 + Bx + A)e^{2x}$.

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7. a) Se boken sid. 480. b) *i)* divergent ty $a_n \equiv 3^{4n+1} \not\to 0$, *ii)* $\sum_{n=1}^\infty \frac{1}{3^{4n+1}} = \frac{1}{3}\sum_{n=1}^\infty(\frac{1}{3^4})^n = \frac{1}{3}(\sum_{n=0}^\infty(\frac{1}{3^4})^n - 1) = (\text{geometrisk serie}) = \frac{1}{3}(\frac{1}{1-\frac{1}{3^4}} - 1) = \frac{1}{240}$, *iii)* $\sum_{n=0}^\infty \frac{1}{n!} = e^1 = e$.

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8. a) $\cos x - 1 = \int_0^x -\sin t\,dt = [\text{PI}] = -([(t-x)\sin t]_0^x - \int_0^x (t-x)\cos t\,dt) = \int_0^x (t-x)\cos t\,dt = [\frac{1}{2}(t-x)^2\cos t]_0^x + \frac{1}{2}\int_0^x (t-x)^2\sin t\,dt = 0 - \frac{x^2}{2} + \frac{1}{2}([\frac{1}{3}(t-x)^3\sin t]_0^x - \frac{1}{3}\int_0^x (t-x)^3\cos t\,dt) = -\frac{x^2}{2} - \frac{1}{6}\int_0^x (t-x)^3\cos t\,dt \equiv -\frac{x^2}{2} + R(x)$ där $|R(x)| \le \frac{1}{6}|\int_0^x |(x-t)^3\cos t|\,dt| \le \frac{1}{6}|\int_0^x |x-t|^3\,dt| \le \frac{1}{6}|x|^3|\int_0^x dt| = \frac{1}{6}|x|^4$. Vi har alltså att $R(x) = \mathcal{O}(x^4)(x)$ och därmed $\cos x = 1 - \frac{x^2}{2} + \mathcal{O}(x^4)$, b) $\int_x^{2x}\frac{\cos t}{t}\,dt = (\text{enl. a)}) = \int_x^{2x}\frac{1 - \frac{t^2}{2} + \mathcal{O}(t^4)}{t}\,dt = \int_x^{2x}\frac{1}{t} + \mathcal{O}(t)\,dt = \int_x^{2x}\mathcal{O}(t)\,dt + [\ln|t|]_x^{2x} = \ln|2x| - \ln|x| + \int_x^{2x}\mathcal{O}(t)\,dt = \ln 2 + \int_x^{2x} t\frac{\mathcal{O}(t)}{t}\,dt$ där $|\int_x^{2x} t\frac{\mathcal{O}(t)}{t}\,dt| \le C\int_x^{2x}|t|\,dt \le C|2x||\int_x^{2x}| \le C|x|^2 \to 0$ då $x \to 0$. Alltså får vi $\lim_{x\to 0}\int_x^{2x}\frac{\cos t}{t}\,dt = \ln 2$.