Matematisk Analys i en variabel E, tmv136, 110429

1) a) $\int_0^{\pi/2} x\sin x\,dx = [PI] = \left[x(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} 1\cdot(-\cos x)dx = \int_0^{\pi/2}\cos x\,dx = \left[\sin x\right]_0^{\pi/2} = 1$

b) $\int \frac{1}{e^x+1}dx = \left[\begin{matrix} t=e^x, x=\ln t \\ \frac{dx}{dt}=\frac{1}{t}dt \end{matrix}\right] = \int \frac{1}{t+1}\frac{1}{t}dt = [PB] = \int \frac{A}{t}+\frac{B}{t+1}dt = \int \frac{1}{t}-\frac{1}{t+1}dt = \ln|t| - \ln|t+1| + C$

$= \ln\left|\frac{t}{t+1}\right| + C = \ln\frac{e^x}{e^x+1} + C$

c) $\int \frac{\ln x\cos(5\ln x)}{x}dx = \left[\begin{matrix} t=\ln x \\ \frac{dt}{dx}=\frac{1}{x}dx \end{matrix}\right] = \int t\cos(5t)dt = [PI] = t\frac{\sin 5t}{5} - \int 1\cdot\frac{\sin 5t}{5}dt =$

$= t\frac{\sin 5t}{5} + \frac{\cos 5t}{25} + C = \frac{\ln x\sin(5\ln x)}{5} + \frac{\cos(5\ln x)}{25} + C$

2) IF: $e^{\int -dx} = e^{-x} \Rightarrow \frac{d}{dx}(e^{-x}y) = x^2e^{-x} \Rightarrow e^{-x}y = \int\frac{d}{dx}(e^{-x}y)dx = \int x^2e^{-x}dx = [PI] =$

$= x^2(-e^{-x}) - \int 2x(-e^{-x})dx = [PI] = -x^2e^{-x} + 2\left(x(-e^{-x}) - \int 1\cdot(-e^{-x})dx\right) = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$

$= -(x^2+2x+2)e^{-x} + C \Rightarrow y = -(x^2+2x+2) + Ce^x$ och $y(0)=1 \Rightarrow y = -(x^2+2x+2) + 3e^x$

3) $y = y_p + y_h$ Kar ekv: $0 = r^2+4r+5 \Rightarrow r_{1,2} = -2\pm i \Rightarrow y_h = e^{-2x}(A\cos x + B\sin x)$

Ansatt $y_p = Ce^x$ insättning $\Rightarrow Ce^x + 4Ce^x + 5Ce^x = 10e^x \Rightarrow C = 1$ $\therefore y = e^x + e^{-2x}(A\cos x + B\sin x)$

$y(0)=2, y'(0)=0 \Rightarrow 2 = y(0) = 1+A,\ 0 = y'(0) = 1-2A+B \Rightarrow y = e^x + e^{-2x}(\cos x + \sin x)$

4) Area $= \int_0^1 x - x^2dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} - 0 = \frac{1}{6}$ Rotationsvolym $= \int_0^1 \pi(x-x^2)^2dx = \pi\int_0^1 x^4 - 2x^3 + x^2 dx = \pi/30$

5) Låt $ds$ vara ett längdelement längs kurvan. Massan i detta längdelement är $dm = \rho(t)ds(t)$ där $\rho$ är densiteten. $\therefore$ enl Pythagoras $ds^2 = dx^2 + dy^2 \Rightarrow$

$\frac{ds}{dt} = \frac{\sqrt{(dx)^2+(dy)^2}}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \Rightarrow ds = \sqrt{(x')^2+(y')^2}dt$

$\therefore$ Massan $= \int_0^1 t\,ds = \int_0^1 t\sqrt{(x')^2+(y')^2}dt = \int_0^1 t\sqrt{2e^{2t}}dt = \sqrt{2}\int_0^1 te^t dt = [PI] = \sqrt{2}\left([te^t]_0^1 - \int_0^1 e^t dt\right) =$

$= \sqrt{2}(e-(e-1)) = \underline{\sqrt{2}}$ ty $x'(t)^2 + y'(t)^2 = (e^t(\cos t - \sin t))^2 + (e^t(\cos t + \sin t))^2 =$

$= e^{2t}(\cos^2 t + \sin^2 t - 2\cos t\sin t + \cos^2 t + \sin^2 t + 2\cos t\sin t) = 2e^{2t}$

6) $\sin^2 x(1-e^{x^2}) = \left(\cancel{x} - \frac{x^3}{3!} + O(x^5)\right)^2\left(1-(1+x^2+O(x^4))\right) = \left(x^2 - \frac{x^4}{3!} - \frac{x^4}{3!} + O(x^6)\right)\left(-x^2 + O(x^4)\right) =$

$= \left(x^2 - \frac{x^4}{3} + O(x^6)\right)\left(-x^2 + O(x^4)\right) = -x^4 + O(x^6)$ Också $(1-\cos x)\ln(1+x) - \frac{x^3}{2} =$

$= \left(\frac{x^2}{2!} + O(x^4)\right)\left(x - \frac{x^2}{2} + O(x^3)\right) - \frac{x^3}{2} = \frac{x^3}{2} - \frac{x^4}{4} + O(x^5) - \frac{x^3}{2} = -\frac{x^4}{4} + O(x^5)$

$\therefore$ Gränsvärdet $= \lim_{x\to 0}\frac{-\frac{x^4}{4} + O(x^5)}{-x^4 + O(x^6)} = \lim_{x\to 0}\frac{\cancel{x^4}(-\frac{1}{4} + O(x))}{\cancel{x^4}(-1 + O(x^2))} = \lim_{x\to 0}\frac{-\frac{1}{4} + O(x)}{-1 + O(x^2)} = \frac{-\frac{1}{4}+0}{-1+0} = \frac{1}{4}$

7) $\int_2^\infty \frac{1}{x(x-1)}dx \le \int_2^\infty \frac{1}{(x-1)^2}dx = \left[\begin{matrix} t=x-1 \\ dt=dx \end{matrix}\right] = \int_1^\infty \frac{1}{t^2}dt < \infty$. Vi har $\int_2^\infty \frac{1}{x(x-1)}dx = \lim_{R\to\infty}\int_2^R \frac{1}{x-1} - \frac{1}{x}dx =$

$= \lim_{R\to\infty}\left[\ln|x-1| - \ln|x|\right]_2^R = \lim_{R\to\infty}\ln\frac{R-1}{R} - (\ln 1 - \ln 2) = \ln\left(\lim_{R\to\infty}\frac{R-1}{R}\right) - 0 + \ln 2 = \ln 1 + \ln 2 = \underline{\ln 2}$

8) Se kurslitteraturen.