Lösningar TMVB1, Envariabel E1, 130405

1) a) $\int x^3(1-x)\,dx = \int x^3 - x^4\,dx = \frac{x^4}{4} - \frac{x^5}{5} + C$ b) $\int \frac{1}{x^3-3x^2+2x}\,dx = \int \frac{1}{x(x-1)(x-2)}\,dx = \{PBU\} = \int \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}\,dx$

$= \int \frac{1/2}{x} - \frac{1}{x-1} + \frac{1/2}{x-2}\,dx = \frac{1}{2}\ln|x| - \ln|x-1| + \frac{1}{2}\ln|x-2| + C$ c) $\int_0^1 \frac{\sqrt{x}}{1+x}\,dx = \begin{bmatrix} t=\sqrt{x} & 0\to 0 \\ x=t^2 & 1\to 1 \\ dx = 2t\,dt & \end{bmatrix} = \int_0^1 \frac{t}{1+t^2}\,2t\,dt =$

$= 2\int_0^1 \frac{t^2}{t^2+1}\,dt = 2\int_0^1 1 - \frac{1}{t^2+1}\,dt = 2\Big[t - \arctan t\Big]_0^1 = 2\left(1 - \frac{\pi}{4}\right) = 2 - \frac{\pi}{2}$

2) $y' + 2xy = 2x$ IF: $e^{\int 2x\,dx} = e^{x^2}$ $\therefore$ $\frac{d}{dx}(e^{x^2}y) = 2xe^{x^2} \Rightarrow e^{x^2}y = \int \frac{d}{dx}(e^{x^2}y)\,dx = \int 2xe^{x^2}\,dx =$

$= e^{x^2} + C \Rightarrow y = 1 + Ce^{-x^2}$ och $2 = y(0) = 1 + Ce^0 = 1 + C \Rightarrow C = 1 \Rightarrow y = 1 + e^{-x^2} \Rightarrow \lim_{x\to\infty} y = 1$

3) $y'' - 4y' + 3y = e^{2x}$ $y = y_p + y_h$ y_h: kar. ekv: $0 = r^2 - 4r + 3 = (r-1)(r-3) \Rightarrow$

$y_h = c_1e^x + c_2e^{3x}$ Ansätt $y_p = x^m Ce^{2x} = \{m=0\} = Ce^{2x} \Rightarrow y_p' = 2Ce^{2x}$, $y_p'' = 4Ce^{2x}$

Insättning $\Rightarrow e^{2x} = y_p'' - 4y_p' + 3y_p = (4 - 4\cdot 2 + 3)Ce^{2x} = -Ce^{2x} \Rightarrow C = -1$ $\therefore$ $y = y_p + y_h =$

$= -e^{2x} + c_1e^x + c_2e^{3x}$, $y' = -2e^{2x} + c_1e^x + 3c_2e^{3x}$ och $y(0) = y'(0) = 0 \Rightarrow -1 + c_1 + c_2 = 0$, $-2 + c_1 + 3c_2 = 0$

$\Rightarrow c_1 = c_2 = 1/2 \Rightarrow$ sökt lösning: $y = -e^{2x} + \frac{1}{2}e^x + \frac{1}{2}e^{3x}$

4) $\gamma(0) = (0, 1/3)$, $\gamma(2) = (2, 5^{3/2})$ Avståndet mellan dessa punkter är $L = \int_0^2 \sqrt{t^2 + \left(\frac{1}{2}(2t+1)^{1/2}\cdot 2\right)^2}\,dt$

$= \int_0^2 \sqrt{t^2+2t+1}\,dt = \int_0^2 \sqrt{(t+1)^2}\,dt = \int_0^2 |t+1|\,dt = \int_0^2 t+1\,dt = \left[\frac{t^2}{2} + t\right]_0^2 = 4$

5) Maclaurinutv.: $\cos x - 1 = 1 - \frac{x^2}{2!} + O(x^4) - 1 = -\frac{x^2}{2} + O(x^4)$ så att $(1-\cos x)^2 =$

$= (\cos x - 1)^2 = \left(-\frac{x^2}{2} + O(x^4)\right)^2 = \frac{x^4}{4} - 2\frac{x^2}{2}O(x^4) + O(x^4)^2 = \frac{x^4}{4} + O(x^6)$ Vidare är

$x(\sin x - x) = x\left(x - \frac{x^3}{3!} + O(x^5) - x\right) = -\frac{x^4}{3!} + O(x^6)$ $\therefore$ $\lim_{x\to 0} \frac{x(\sin x - x)}{(1-\cos x)^2} = \lim_{x\to 0} \frac{\cancel{x^4}\left(-\frac{1}{6} + O(x^2)\right)}{\cancel{x^4}\left(\frac{1}{4} + O(x^2)\right)} = \frac{-\frac{1}{6} + 0}{\frac{1}{4} + 0}$

$= -\frac{2}{3}$

6) Vi har $y(0) = 1 + \int_0^0 t\,y(t)\,dt = 1 + 0 = 1$ Derivering ger $y' = 0 + \frac{d}{dx}\int_0^x t\,y(t)\,dt =$

$= xy(x) \Rightarrow y' - xy = 0$ IF: $e^{\int -x\,dx} = e^{-x^2/2}$ $\therefore$ $\frac{d}{dx}\left(e^{-x^2/2}y\right) = e^{-x^2/2}\cdot 0 = 0 \Rightarrow$

$e^{-x^2/2}y(x) = \int \frac{d}{dx}\left(e^{-x^2/2}y\right)dx = \int 0\,dx = C$ $\therefore$ $y = Ce^{x^2/2}$ och $y(0) = 1 \Rightarrow$

$C = 1$ $\therefore$ $y = e^{x^2/2}$

7) Låt $y_1 = y$, $y_2 = y_1' = y'$, $y_3 = y_2' = y''$, $y_3' = y''' = g(x) - \cos x\, y_3 - 2\sin x\, y_2 + x y_1$

Låt $\bar{Y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ Då gäller $\bar{Y}' = \begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ g - \cos x\, y_3 - 2\sin x\, y_2 + x y_1 \end{pmatrix} =$

$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x & -2\sin x & -\cos x \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ g(x) \end{pmatrix} \equiv A\bar{Y} + B$

8) Låt $x_0 \in I$ och $x \in I$ vara godtyckliga. Medelvärdessatsen $\Rightarrow$

$y(x) - y(x_0) = y'(\xi)(x - x_0)$ där ξ är en punkt mellan x_0 och x

Då $y'(x) = 0$ på I får vi $y(x) - y(x_0) = 0(x - x_0) = 0 \Rightarrow$

$y(x) = y(x_0)$, $\forall x \in I$.