Analys i en variabel, E, TMV137 150413

1) a) $\int \cos(2x)\,dx = \frac{\sin(2x)}{2} + C$

b) $\int_0^{\pi^2/4} \sin\sqrt{x}\,dx = \left[\begin{array}{l} t=\sqrt{x} \text{ och} \\ x=t^2 \quad \frac{\pi^2}{4} \Rightarrow \frac{\pi}{2} \\ \frac{dx}{dt}=2t\,dt \end{array}\right.$

$= \int_0^{\pi/2} (\sin t)\,2t\,dt = [PI] = 2\left(\left[t(-\cos t)\right]_0^{\pi/2} - \int_0^{\pi/2} 1\cdot(-\cos t)\,dt\right)$

$= 2\left(0 + \int_0^{\pi/2} \cos t\,dt\right) = 2\left[\sin t\right]_0^{\pi/2} = 2(1-0) = 2$

c) $x^2-x-2 \,\overline{)\, x^3-3x-1}$ with quotient $x+1$:
$-(x^3-x^2-2x)$
x^2-x-1
$-(x^2-x-2)$
1

$\Rightarrow \int \frac{x^3-3x-1}{x^2-x-2}\,dx = \int x+1+\frac{1}{x^2-x-2}\,dx$

$= \frac{x^2}{2}+x+\int\frac{1}{(x+1)(x-2)}\,dx =$

$= \frac{x^2}{2}+x+\int \frac{-1/3}{x+1}+\frac{1/3}{x-2}\,dx = \frac{x^2}{2}+x-\frac{1}{3}\left(\ln|x+1|-\ln|x-2|+C\right) =$

$= \frac{x^2}{2}+x-\frac{1}{3}\ln\left|\frac{x+1}{x-2}\right| + C$

2) Låt $f(x) = \arcsin x$. Då gäller $f(0)=0$, $f'(x) = (1-x^2)^{-1/2}$ och $f'(0)=1$, $f''(x) = x(1-x^2)^{-3/2}$ och $f''(0)=0$, $f^{(3)}(x) = (1-x^2)^{-3/2} + 3x^2(1-x^2)^{-5/2}$ och $f'''(0)=1$ $\therefore$ Maclaurin: $f(x) = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\mathcal{O}(x^4)$

$= x+\frac{1}{6}x^3+\mathcal{O}(x^4) = x+\mathcal{O}(x^3) \Rightarrow x\arcsin x = x^2(1+\mathcal{O}(x^2))$

$e^x = 1+x+\frac{x^2}{2}+\mathcal{O}(x^3)$, $\cos x = 1-\frac{x^2}{2!}+\mathcal{O}(x^4)$

$\therefore \frac{(e^x-\cos x)^2}{x\arcsin x} = \frac{\left(1+x+\mathcal{O}(x^2)-(1-\mathcal{O}(x^2))\right)^2}{x^2(1+\mathcal{O}(x^2))} = \frac{(x+\mathcal{O}(x^2))^2}{x^2(1+\mathcal{O}(x^2))} =$

$= \frac{\cancel{x^2}(1+\mathcal{O}(x))^2}{\cancel{x^2}(1+\mathcal{O}(x^2))} = \frac{(1+\mathcal{O}(x))^2}{1+\mathcal{O}(x^2)} \to \frac{(1+0)^2}{1+0} = \frac{1}{1} = 1$ då $x\to 0$.

3) Den allm. lösn. har lösning $y = y_p + y_h$

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För y_h: Kar. ekv. $0 = r^2 - 2r + 1 = (r-1)^2 \Rightarrow r_{1,2} = 1 \Rightarrow$

$y_h = (Ax+B)e^x$

För y_p: Ansätt $y_p = x^m C e^x =$ ($m = 2$ annars $y_p \in y_h$)

$= Cx^2e^x \Rightarrow y_p' = C(2xe^x + x^2e^x) = 2Cxe^x + y_p \Rightarrow$

$y_p'' = D(y_p') = 2C(e^x + xe^x) + y_p' = 2C(1+x)e^x +$

$+ 2Cxe^x + y_p$. Insättning i DE $\Rightarrow$

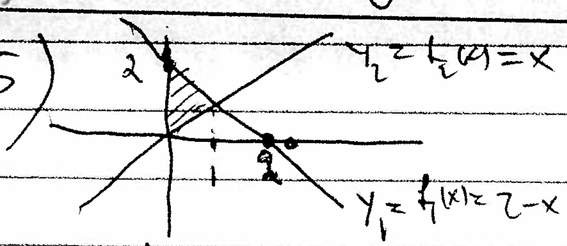
$e^x = y_p'' - 2y_p' + y_p = 2C(1+2x)e^x + y_p - 2(2Cxe^x + y_p) + y_p =$

$= (2C(1+2x) - 4Cx)e^x = 2Ce^x \Rightarrow 2C = 1 \Rightarrow C = \frac{1}{2}$

$\therefore\ y = y_p + y_h = \frac{1}{2}x^2e^x + (Ax+B)e^x = \left(\frac{1}{2}x^2 + Ax + B\right)e^x$

4) Se föregående tenta, 150114, uppg. 4)

5)



Rotation runt x-axeln: $A =$

$= \int_0^1 2\pi f_1(x)\,ds_1 + \int_0^1 2\pi f_2(x)\,ds_2 + \pi(2)^2 =$

$= 2\pi\int_0^1 (2-x)\sqrt{1+(-1)^2}\,dx + 2\pi\int_0^1 x\sqrt{1+1^2}\,dx + 4\pi = 4\sqrt{2}\pi\int_0^1 dx + 4\pi =$

$= 4\sqrt{2}\pi + 4\pi = \underline{\underline{4(1+\sqrt{2})\pi}}.$

Rotation runt y-axeln: $A =$ (symmetriskäl) $= 2\int_0^1 2\pi x\,ds_1 =$

$= 4\pi\int_0^1 x\sqrt{1+(-1)^2}\,dx = 4\sqrt{2}\pi\int_0^1 x\,dx = 4\sqrt{2}\pi\left[\frac{x^2}{2}\right]_0^1 = \underline{\underline{2\sqrt{2}\pi}}$

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6) Derivatan av likheten $\Rightarrow$ (Obs, $x>0$ pga $\ln x$)

$$xy + \frac{x^2}{2}y' + e^{2\ln x}\, y(e^{\ln x})\cdot\frac{1}{x} = 2x$$

$$\Leftrightarrow \quad y' + \frac{4}{x}y = \frac{4}{x}, \quad \text{Linj. ekv}$$

$$\text{IF: } e^{\int \frac{4}{x}dx} = e^{4\ln x} = e^{\ln x^4} = x^4 \Rightarrow$$

$$\frac{d}{dx}(x^4 y) = 4x^3 \Rightarrow x^4 y = \int \frac{d}{dx}(x^4 y)\,dx = \int 4x^3 dx = x^4 + C$$

$\therefore$ $y = 1 + Cx^{-4}$. Insättning $x = e$ i integralekv. $\Rightarrow$

$$\frac{e^2}{2}y(e) + \int_1^{\ln e} e^{2t} y(e^t)\,dt = e^2 \Leftrightarrow \frac{e^2}{2}y(e) + 0 = e^2 \Leftrightarrow$$

$$y(e) = 2 \quad \therefore \quad y(e) = 1 + Ce^{-4} = 2 \Rightarrow C = e^4$$

$$\therefore \quad y(x) = 1 + e^4 x^{-4} = 1 + \left(\frac{e}{x}\right)^4$$

7) Vi har $(k+1)^3 - k^3 = k^3 + 3k^2 + 3k + 1 - k^3 = 3k^2 + 3k + 1$

Summation $\Rightarrow$ $(n+1)^3 - 1^3 = (n+1)^3 + \cancel{n^3} + (n-1)^3 + \dots + 2^3 - (n^3 + (n-1)^3 + \dots + 2^3 + 1^3)$

$$= \sum_{k=1}^{n}(k+1)^3 - \sum_{k=1}^{n} k^3 = \sum_{k=1}^{n}(k+1)^3 - k^3 = \sum_{k=1}^{n} 3k^2 + 3k + 1 =$$

$$= 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = 3\sum_{k=1}^{n} k^2 + 3\frac{n(n+1)}{2} + n$$

$$\therefore \quad 3\sum_{k=1}^{n} k^2 = (n+1)^3 - 1 - \frac{3n(n+1)}{2} - n = \frac{2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n}{2}$$

$$= \frac{2n^3 + 3n^2 + n}{2} = \frac{n(2n^2 + 3n + 1)}{2} = \frac{n(n+1)(2n+1)}{2}$$

$$\therefore \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

8) Lös $f(x) = \begin{cases} 0 & x \in [0,1] \quad x \text{ rationell} \\ 1 & x \in [0,1] \quad x \text{ irrationell.} \end{cases}$ $\quad \int_0^1 f(x)dx \; \nexists$