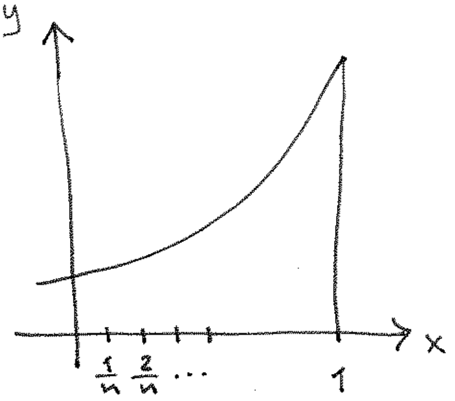
Envariabelsanalys Z & TD, ht2017, Föreläsning 7.1

Ex. Låt $f(x) = e^x$ och beräkna $\int_0^1 f(x)\,dx$ genom att visa att $\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} U(f,P_n)$ där P_n är $[0,1]$ indelad i n st delintervall av längd $\frac{1}{n}$.



Lösn.: För varje delintervall

$$[x_{j-1}, x_j] = \left[\frac{j-1}{n}, \frac{j}{n}\right],\ j = 1, \ldots, n$$

gäller att

$$m_j = \min_j f = f\left(\frac{j-1}{n}\right) = e^{\frac{j-1}{n}}$$

$$M_j = \max_j f = f\left(\frac{j}{n}\right) = e^{j/n}$$

$$\Rightarrow L(f,P_n) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}) = \sum_{j=1}^{n} e^{\frac{j-1}{n}} \cdot \frac{1}{n} =$$

$$= \left\{ \begin{matrix} k = j-1, & j=1 \Leftrightarrow k=0 \\ j = k+1, & j=n \Leftrightarrow k=n-1 \end{matrix} \right\} = \sum_{k=0}^{n-1} \frac{1}{n} e^{k/n} =$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} (e^{1/n})^k = \frac{1}{n} \cdot \frac{(e^{1/n})^n - 1}{e^{1/n} - 1} = (e-1) \cdot \frac{1/n}{e^{1/n} - 1} \xrightarrow[n\to\infty]{(*)} e - 1$$

$$\lim_{n\to\infty} \frac{1/n}{e^{1/n} - 1} = \lim_{x\to 0} \frac{x}{e^x - 1} = \{\text{l'Hospital}\} = \lim_{x\to 0} \frac{1}{e^x} = 1 \qquad (*)$$

$$U(f,P_n) = \sum_{j=1}^{n} M_j(x_j - x_{j-1}) = \sum_{j=1}^{n} e^{j/n} \cdot \frac{1}{n} = \frac{1}{n}\sum_{j=1}^{n} (e^{1/n})^j =$$

$$= \frac{1}{n}\left(\sum_{j=0}^{n} (e^{\frac{1}{n}})^j - 1\right) = \frac{1}{n}\left(\frac{(e^{1/n})^{n+1} - 1}{e^{1/n} - 1} - 1\right) =$$

$$= \frac{1}{n} \cdot \frac{e \cdot e^{1/n} - \cancel{1} - e^{1/n} + \cancel{1}}{e^{1/n} - 1} = (e-1)\, \frac{\frac{1}{n} \cdot e^{1/n}}{e^{1/n} - 1} \xrightarrow[n\to\infty]{(**)} e - 1$$

$$\lim_{n\to\infty} \frac{1/n \, e^{1/n}}{e^{1/n}-1} = \lim_{x\to 0} \frac{xe^x}{e^x-1} = \lim_{x\to 0} \frac{e^x + xe^x}{e^x} = 1 \qquad (**)$$

$$\therefore \int_0^1 e^x \, dx = e - 1$$

<u>Ex</u>. Visa att $1 \leqslant \int_1^4 \frac{1}{1+\sqrt{x}} \, dx \leqslant \frac{3}{2}$ utan att beräkna integralen.

<u>Lösn.</u>: Låt $f(x) = \frac{1}{1+\sqrt{x}} = (1+\sqrt{x})^{-1}$

$$f'(x) = -\frac{1}{(1+\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} < 0 \text{ på } (1,4)$$

$$\Rightarrow f \text{ avtagande på } [1,4] \Rightarrow$$

$$\min_{x\in[1,4]} f(x) = f(4) = \frac{1}{1+\sqrt{4}} = \frac{1}{3}$$

$$\max_{x\in[1,4]} f(x) = f(1) = \frac{1}{1+\sqrt{1}} = \frac{1}{2}$$

$$\therefore \frac{1}{3} \leqslant \frac{1}{1+\sqrt{x}} \leqslant \frac{1}{2} \Rightarrow$$

$$\Rightarrow \int_1^4 \frac{1}{3} \, dx \leqslant \int_1^4 \frac{1}{1+\sqrt{x}} \, dx \leqslant \int_1^4 \frac{1}{2} \, dx \Leftrightarrow$$

$$\Leftrightarrow 1 \leqslant \int_1^4 \frac{1}{1+\sqrt{x}} \, dx \leqslant \frac{3}{2} \quad \blacksquare$$

<u>Ex</u>. Visa att $\lim_{n\to\infty} \int_n^{n+2} \arctan(x) \, dx = \pi$ utan att beräkna integralen.

<u>Bevis</u>: Integralkalkylens medelvärdessats ger att:

$$\exists\, \xi_n \in [n, n+2] \;:\; \int_n^{n+2} \arctan(x)\,dx = \arctan(\xi_n)(\cancel{n}+2-\cancel{n})$$

$$n \leq \xi_n \leq n+2 \quad \text{så } n \to \infty \Rightarrow \xi_n \to \infty$$

$$\therefore \lim_{n\to\infty} \int_n^{n+2} \arctan(x)\,dx = \lim_{n\to\infty} 2\arctan(\xi_n) = 2 \cdot \frac{\pi}{2} = \pi \quad \blacksquare$$

<u>Ex</u>. Beräkna följande integraler/primitiva funktioner:

(a) $\displaystyle\int_0^1 \frac{x}{1+\sqrt{x}}\,dx$ (b) $\displaystyle\int \frac{1}{\sqrt{\sqrt{x}-2}}\,dx$ (c) $\displaystyle\int \frac{1}{\sqrt{7-5x^2}}\,dx$

<u>Lösn.</u>: (a) $\displaystyle\int_0^1 \frac{x}{1+\sqrt{x}}\,dx = \left\{\begin{array}{ll} t = 1+\sqrt{x}, & x=0 \Leftrightarrow t=1 \\ x = (t-1)^2, & x=1 \Leftrightarrow t=2 \\ dx = 2(t-1)\,dt & \end{array}\right\} =$

$$= \int_1^2 \frac{(t-1)^2}{t} 2(t-1)\,dt = 2\int_1^2 \frac{t^3-3t^2+3t+1}{t}\,dt =$$

$$= 2\int_1^2 \left(t^2-3t+3 - \frac{1}{t}\right)dt = 2\left[\frac{t^3}{3} - \frac{3t^2}{2} + 3t - \ln|t|\right]_1^2 =$$

$$= 2\left(\frac{8}{3} - \cancel{6} + \cancel{6} - \ln(2) - \left(\frac{1}{3} - \frac{3}{2} + 3\right)\right) =$$

$$= 2\left(\frac{16-2+9-18}{6} - \ln(2)\right) = \frac{5}{3} - 2\ln(2)$$

(b) $\displaystyle\int \frac{1}{\sqrt{\sqrt{x}-2}}\,dx = \left\{\begin{array}{l} t = \sqrt{x} \\ dx = 2t\,dt \end{array}\right\} = \int \frac{2t}{\sqrt{t-2}}\,dt = \left\{\begin{array}{l} s = \sqrt{t-2} \\ t = 2+s^2 \\ dt = 2s\,ds \end{array}\right\} =$

$$= \int \frac{2(2+s^2)}{\cancel{s}} \cdot 2\cancel{s}\,ds = 4\int (2+s^2)\,ds = 4\left(2s + \frac{s^3}{3}\right) =$$

$$= 4\left(2\sqrt{t-2} + \frac{(t-2)^{3/2}}{3}\right) = 8\sqrt{\sqrt{x}-2} + \frac{4(\sqrt{x}-2)^{3/2}}{3} + C$$

(c) $\int \frac{dx}{\sqrt{7-5x^2}} = \frac{1}{\sqrt{7}} \int \frac{dx}{\sqrt{1-\left(\sqrt{\frac{5}{7}}x\right)^2}} = \frac{1}{\sqrt{7}} \cdot \sqrt{\frac{7}{5}} \arcsin\left(\sqrt{\frac{5}{7}}x\right) =$

$= \frac{1}{\sqrt{5}} \arcsin\left(\sqrt{\frac{5}{7}}x\right) + C$

Ex. Beräkna alla primitiva funktioner till $f(x) = \frac{2x^5+x^4}{x^4-1}$

Lösn.: $\frac{2x^5+x^4}{x^4-1}$ ← grad = 5, ← grad = 4 $\Rightarrow$ polynomdivision

$$\begin{array}{r|l} & 2x+1 \\ \hline x^4-1 & 2x^5+x^4 \\ & -(2x^5-2x) \\ \hline & x^4+2x \\ & -(x^4-1) \\ \hline & 2x+1 \end{array}$$

$\Rightarrow \int f(x)\,dx = \int \left(2x+1+\frac{2x+1}{x^4-1}\right) dx =$

$= x^2 + x + \int \frac{2x+1}{(x^2+1)(x+1)(x-1)}\,dx$

Partialbråksuppdelning:

$\frac{2x+1}{(x^2+1)(x+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1} \Leftrightarrow$

$2x+1 = (Ax+B)(x^2-1) + C(x^2+1)(x-1) + D(x^2+1)(x+1)$

$\underline{x=1}$: $3 = 4D \Leftrightarrow D = 3/4$

$\underline{x=-1}$: $-1 = -4C \Leftrightarrow C = 1/4$

$Ax^3+Bx^2-Ax-B = 2x+1-\frac{1}{4}(x^2+1)(x-1) - \frac{3}{4}(x^2+1)(x+1)$

$$Ax^3 + Bx^2 - Ax - B = 2x + 1 - \frac{1}{4}(4x^3 + 2x^2 + 4x + 2)$$

$$\Rightarrow \quad A = -1 \ , \ B = -\frac{1}{2}$$

$$\therefore \int f(x)\,dx = x^2 + x + \int \left(\frac{-x - 1/2}{x^2+1} + \frac{1}{4}\cdot\frac{1}{x+1} + \frac{3}{4}\cdot\frac{1}{x-1} \right) dx =$$

$$= x^2 + x - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx - \frac{1}{2}\int \frac{1}{x^2+1}\,dx + \frac{1}{4}\ln|x+1| + \frac{3}{4}\ln|x-1| =$$

$$= x^2 + x - \frac{1}{2}\ln(x^2+1) - \frac{1}{2}\arctan(x) + \frac{1}{4}\ln|x+1| + \frac{3}{4}\ln|x-1| + C$$