Envariabelsanalys Z & TD, ht2017, Föreläsning 7.2

Ex. Lös följande ODE:n / BVP

(a) $\begin{cases} y' = x\tan(y) \\ y(0) = \frac{\pi}{6} \end{cases}$ (b) $y' - y\cdot\tan(x) = \cos(x)$, $|x| < \frac{\pi}{2}$

(c) $\begin{cases} y'' + y = 2\sin^2\left(\frac{x}{2}\right) \\ y(0) = 1,\ y'(0) = 0 \end{cases}$

Lösn.: (a) $y = x\cdot\tan(y) \Leftrightarrow \frac{1}{\tan(y)}y' = x \Rightarrow$

$$\Rightarrow \int \frac{1}{\tan(y)}\,dy = \int x\,dx = \frac{1}{2}x^2 + C$$

$$\int \frac{1}{\tan(y)}\,dy = \int \frac{\cos(y)}{\sin(y)}\,dy = \left\{ \begin{array}{l} t = \sin(y) \\ dt = \cos(y)\,dy \end{array} \right\} =$$

$$= \int \frac{1}{t}\,dt = \ln|t| = \ln|\sin(y)|$$

$$\Rightarrow \ln|\sin(y)| = \frac{1}{2}x^2 + C \Leftrightarrow \sin(y) = \pm e^c\cdot e^{x^2/2} = Ce^{x^2/2}$$

$y(0) = \frac{\pi}{6}$: $\sin\left(\frac{\pi}{6}\right) = C\cdot e^0 \Leftrightarrow C = \frac{1}{2}$

$$\therefore\ y(x) = \arcsin\left(\frac{e^{x^2/2}}{2}\right) \text{ def. för } \frac{e^{x^2/2}}{2} \le 1 \Leftrightarrow |x| \le \sqrt{\ln(4)}$$

(b) $y' - \tan(x)\cdot y = \cos(x)$ linjär

$$\int -\tan(x)\,dx = \int \frac{-\sin(x)}{\cos(x)}\,dx = \ln|\cos(x)| \overset{|x|<\frac{\pi}{2}}{=} \ln(\cos(x))$$

$$\Rightarrow \text{Integr. faktor}: e^{\ln(\cos(x))} = \cos(x)$$

$$\Rightarrow \frac{d}{dx}\left(\cos(x)y\right) = \cos^2(x) \Rightarrow$$

$$\Rightarrow \cos(x)\,y(x) = \int \cos^2(x)\,dx = \left\{ \begin{array}{l} \cos(2x) = \cos^2(x) - \sin^2(x) = \\ = 2\cos^2(x) - 1 \end{array} \right\} =$$

$$= \int \frac{1+\cos(2x)}{2}\,dx = \frac{x}{2} + \frac{\sin(2x)}{4} + C$$

$$\therefore\ y(x) = \frac{x}{2\cos(x)} + \frac{\sin(2x)}{4\cos(x)} + \frac{C}{\cos(x)} =$$

$$= \frac{x}{2\cos(x)} + \frac{\sin(x)}{2} + \frac{C}{\cos(x)}$$

(c) (i) <u>Homogenlösn.</u>: Kar. ekv. : $r^2 + 1 = 0 \Rightarrow r = \pm i$

$$\Rightarrow y_h(x) = A\cos(x) + B\sin(x)$$

(ii) <u>Partikulärlösn.</u>: $\cos(2t) = 1 - 2\sin^2(t) \Leftrightarrow$

$$\Leftrightarrow 2\sin^2(t) = 1 - \cos(2t) \overset{t=\frac{x}{2}}{\Rightarrow} 2\sin^2\left(\frac{x}{2}\right) = 1 - \cos(x)$$

$$\therefore\ y'' + y = 2\sin^2\left(\frac{x}{2}\right) \Leftrightarrow y'' + y = 1 - \cos(x)$$

Studera hjälpekv. :

I. $u'' + u = 1 \Rightarrow u_p = 1 = y_{p_1}$

II. $v'' + v = e^{ix}$. Låt $v = ze^{ix} \Rightarrow$

$$\Rightarrow v' = z'e^{ix} + ize^{ix},\quad v'' = z''e^{ix} + 2iz'e^{ix} - ze^{ix}$$

$$\Rightarrow v'' + v = (z'' + 2iz')e^{ix} \overset{\text{vill}}{=} e^{ix} \Leftrightarrow z'' + 2iz' = 1$$

$$\Rightarrow z_p = \frac{x}{2i} = -\frac{i}{2}x \Rightarrow v_p = z_p e^{ix} =$$

$$= -\frac{i}{2}x(\cos(x) + i\sin(x)) = \frac{x}{2}\sin(x) - i\frac{x}{2}\cos(x)$$

$$\Rightarrow y_{p_2} = -\operatorname{Re}(v_p) = -\frac{x}{2}\sin(x)$$

$$y_p = y_{p_1} + y_{p_2} = 1 - \frac{x}{2}\sin(x)$$

$$\therefore y = y_h + y_p = A\cos(x) + B\sin(x) + 1 - \frac{x}{2}\sin(x)$$

(iii) <u>Begynnelsevillkoren:</u>

$$y(0) = A + 1 = 1 \iff A = 0$$

$$\Rightarrow y'(x) = B\cos(x) - \frac{1}{2}\sin(x) - \frac{x}{2}\cos(x)$$

$$y'(0) = B = 0$$

$$\therefore y(x) = 1 - \frac{x}{2}\sin(x)$$

<u>Ex.</u> Bestäm en deriverbar funktion $y(x)$ definierad $\forall x \in \mathbb{R}$ som uppfyller $\begin{cases} y' = 3y^{2/3} \\ y(0) = -1,\ y(3) = 1 \end{cases}$

<u>Lösn.:</u> $y' = 3y^{2/3} \iff \dfrac{1}{3y^{2/3}}\, y' = 1 \Rightarrow$

$$\Rightarrow \int \frac{y^{-2/3}}{3}\, dy = \int 1\, dx = x + C$$

$$\int \frac{y^{-2/3}}{3}\, dy = y^{1/3} \Rightarrow y^{1/3} = x + C$$

<u>$y(0) = -1$:</u> $(-1)^{1/3} = 0 + C \iff C = -1$

<u>$y(3) = 1$:</u> $(1)^{1/3} = 3 + C \iff C = -2$

Vi vill alltså hitta en funktion som är deriverbar

$\forall x \in \mathbb{R}$ som är $(x-1)^3$ nära $x=0$ och $(x-2)^3$ nära $x=3$.

Ser att $y(x)=0$ $\forall x \in \mathbb{R}$ också är en lösning till $y' = 3y^{2/3}$.

Vi vill alltså pussla ihop $(x-1)^3$, 0, och $(x-2)^3$ till en deriverbar funktion y med $y(0)=-1$ och $y(3)=1$.

Låt
$$y(x) = \begin{cases} (x-1)^3 & \text{om } x \leq 1 \\ 0 & \text{om } 1 < x < 2 \\ (x-2)^3 & \text{om } x \geq 2 \end{cases}$$

Både $(x-1)^3$ och $(x-2)^3$ har derivatan 0 vid $x=1$ respektive $x=2$, så y är deriverbar $\forall x \in \mathbb{R}$.

Ex. Låt $\ell > 0$ och bestäm alla $\lambda \in \mathbb{R}$ för vilka
$$\begin{cases} y'' + \lambda y = 0 \\ y'(0) = y'(\ell) = 0 \end{cases}$$
har en icke-trivial lösning $y(x)$.

Lösn.: Karak. ekv.: $r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{-\lambda}$

$\lambda < 0$: $y(x) = C_1 e^{\sqrt{-\lambda}\,x} + C_2 e^{-\sqrt{-\lambda}\,x} \Rightarrow$

$\Rightarrow y'(x) = \sqrt{-\lambda}\, C_1 e^{\sqrt{-\lambda}\,x} - \sqrt{-\lambda}\, C_2 e^{-\sqrt{-\lambda}\,x}$

$$y'(0) = \sqrt{-\lambda}\,(C_1 - C_2) = 0 \iff C_1 = C_2$$

$$y'(\ell) = \sqrt{-\lambda}\, C_1 \underbrace{\left(e^{-\sqrt{-\lambda}\cdot \ell} - e^{-\sqrt{-\lambda}\cdot \ell}\right)}_{>0} = 0 \Rightarrow C_1 = 0$$

$\therefore \nexists$ icke-trivial lösning då $\lambda < 0$

$\underline{\lambda = 0}$: $y(x) = (C_1 x + C_2)e^0 = C_1 x + C_2$

$$y'(x) = C_1 \Rightarrow y'(0) = C_1 = 0 \Rightarrow$$

$$\Rightarrow y(x) = C_2 \text{ konstant}$$

$\therefore \nexists$ icke-trivial lösning då $\lambda = 0$

$\underline{\lambda > 0}$: $\sqrt{-\lambda} = i\sqrt{\lambda} \Rightarrow y(x) = A\cos(\sqrt{\lambda}\cdot x) + B\sin(\sqrt{\lambda}\cdot x)$

$$\Rightarrow y'(x) = -\sqrt{\lambda}\,A\sin(\sqrt{\lambda}\cdot x) + \sqrt{\lambda}\,B\cos(\sqrt{\lambda}\cdot x)$$

$$y'(0) = \sqrt{\lambda}\cdot B = 0 \Rightarrow B = 0$$

$$y'(\ell) = -\sqrt{\lambda}\,A\sin(\sqrt{\lambda}\cdot \ell) = 0 \Rightarrow \sin(\sqrt{\lambda}\cdot\ell) = 0$$

$$\Rightarrow \sqrt{\lambda}\cdot \ell = n\pi \,.\; n \in \mathbb{Z}$$

$$\therefore \lambda = \left(\frac{n\pi}{\ell}\right)^2 ,\; n = 0, 1, 2, \dots$$

med motsvarande lösningar

$$y(x) = A\cos\left(\frac{n\pi x}{\ell}\right)$$