Envariabelsanalys Z & TD, ht2017, Räkneövning 1.1

5.1.19 Beräkna summan $\sum_{k=1}^{n} (\pi^k - 3)$

Lösn.: $\sum_{k=1}^{n}(\pi^k - 3) = \sum_{k=1}^{n} \pi^k - \sum_{k=1}^{n} 3 = \sum_{k=1}^{n} \pi^k - 3n$

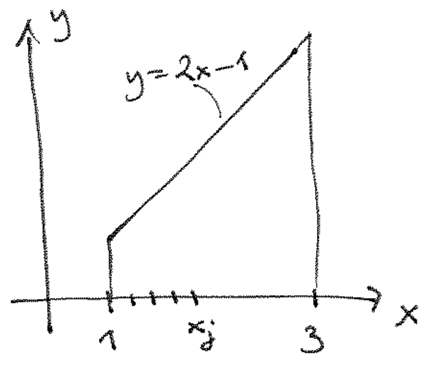
Vet att: $\sum_{j=0}^{n} x^j = 1 + x + x^2 + \dots + x^n = \frac{x^{n+1} - 1}{x - 1}, \; x \neq 1$

$$\Rightarrow \sum_{k=1}^{n} \pi^k = \pi + \pi^2 + \dots + \pi^n = 1 + \pi + \pi^2 + \dots + \pi^n - 1 =$$

$$= \frac{\pi^{n+1} - 1}{\pi - 1} - 1 = \frac{\pi^{n+1} - \cancel{1} - \pi + \cancel{1}}{\pi - 1} = \frac{\pi(\pi^n - 1)}{\pi - 1}$$

$$\therefore \sum_{k=1}^{n} (\pi^k - 3) = \frac{\pi(\pi^n - 1)}{\pi - 1} - 3n$$

5.2.3 Beräkna arean av området under $y = 2x - 1$, över $y = 0$ och mellan $x = 1$ och $x = 3$ genom att approximera med rektanglar.



Lösn.: Vi delar upp $[1, 3]$ i n st delintervall med längd $\frac{3-1}{n} = \frac{2}{n}$

$$\Rightarrow [x_{j-1}, x_j] = \left[1 + \frac{2}{n}(j-1), \; 1 + \frac{2}{n}j\right], \; j = 1, \dots, n$$

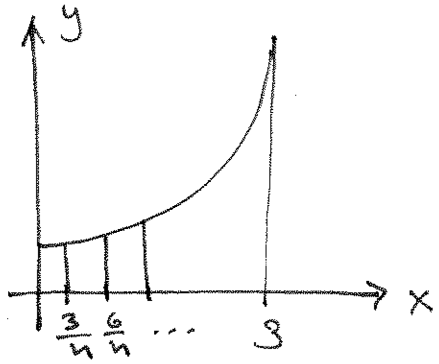
Vi väljer $y\left(1 + \frac{2}{n}j\right) = 2\left(1 + \frac{2}{n}j\right) - 1 = 1 + \frac{4}{n}j$ som höjd i varje delintervall

$$\Rightarrow \sum_{j=1}^{n} \left(1 + \frac{4}{n}j\right) \cdot \frac{2}{n} = \frac{2}{n}\left(\sum_{j=1}^{n} 1 + \frac{4}{n} \sum_{j=1}^{n} j\right) =$$

$$= \frac{2}{n}\left(n + \frac{4}{\cancel{n}} \cdot \frac{\cancel{n}(n+1)}{2}\right) = \frac{2}{n}(3n+2) = 6 + \frac{4}{n}$$

$$\therefore \text{Area} = \lim_{n\to\infty}\left(6 + \frac{4}{n}\right) = 6 \text{ a.e.}$$

5.3.10 Låt $f(x) = e^x$ och beräkna $\int_0^3 f(x)\,dx$ genom att visa att $\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n)$ där P_n är $[0,3]$ indelad i n st delintervall av längd $\frac{3-0}{n} = \frac{3}{n}$



Lösn.: För varje delintervall

$$[x_{j-1}, x_j] = \left[\frac{3}{n}(j-1), \frac{3}{n}j\right], \; j = 1, \ldots, n$$

gäller att:

$$m_j = \min f = f\left(\frac{3}{n}(j-1)\right) = e^{\frac{3}{n}(j-1)}$$

$$M_j = \max f = f\left(\frac{3}{n}j\right) = e^{\frac{3}{n}j}$$

$$\Rightarrow L(f, P_n) = \sum_{j=1}^{n} m_j (x_j - x_{j-1}) = \sum_{j=1}^{n} e^{\frac{3}{n}(j-1)} \cdot \frac{3}{n} =$$

$$= \left\{ \begin{matrix} k = j-1, & j=1 \Leftrightarrow k=0 \\ j = k+1, & j=n \Leftrightarrow k=n-1 \end{matrix} \right\} = \sum_{k=0}^{n-1} \frac{3}{n} e^{\frac{3}{n}k} =$$

$$= \frac{3}{n}\sum_{k=0}^{n-1} (e^{3/n})^k = \frac{3}{n} \frac{(e^{3/n})^n - 1}{e^{3/n} - 1} = (e^3 - 1) \cdot \frac{3/n}{e^{3/n} - 1} \xrightarrow[n\to\infty]{(*)} e^3 - 1$$

$$\lim_{n\to\infty} \frac{3/n}{e^{3/n} - 1} = \lim_{x\to 0} \frac{x}{e^x - 1} = \{\text{l'Hopital}\} = \lim_{x\to 0} \frac{1}{e^x} = 1 \quad (*)$$

$$U(f, P_n) = \sum_{j=1}^{n} M_j (x_j - x_{j-1}) = \sum_{j=1}^{n} e^{\frac{3}{n}j} \cdot \frac{3}{n} = \frac{3}{n}\sum_{j=1}^{n} (e^{3/n})^j =$$

$$= \frac{3}{n} \cdot \left(\sum_{j=0}^{n} (e^{3/n})^j - 1\right) = \frac{3}{n} \cdot \left(\frac{(e^{3/n})^{n+1} - 1}{e^{3/n} - 1} - 1\right) =$$

$$= \frac{3}{n} \cdot \frac{e^3 \cdot e^{3/n} - \cancel{1} - e^{3/n} + \cancel{1}}{e^{3/n} - 1} = (e^3 - 1) \cdot \frac{3/n \cdot e^{3/n}}{e^{3/n} - 1} \xrightarrow[n \to \infty]{(**)} e^3 - 1$$

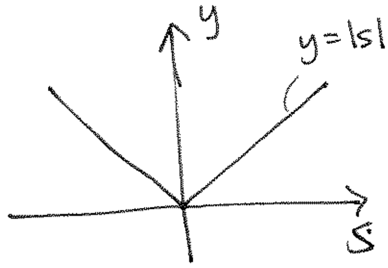
$$\lim_{n \to \infty} \frac{3/n \cdot e^{3/n}}{e^{3/n} - 1} = \lim_{x \to 0} \frac{x e^x}{e^x - 1} = \{\text{l'Hopital}\} =$$

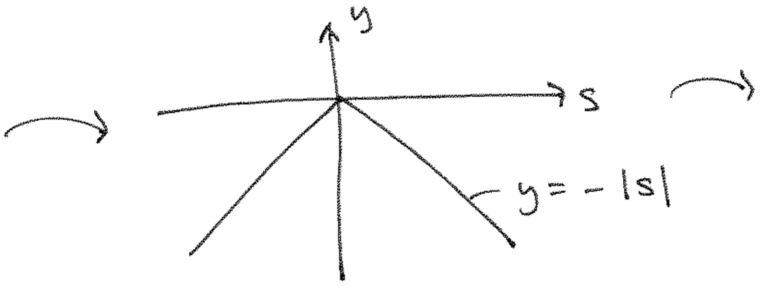
$$= \lim_{x \to 0} \frac{e^x + x e^x}{e^x} = \frac{1 + 0}{1} = 1 \qquad (**)$$

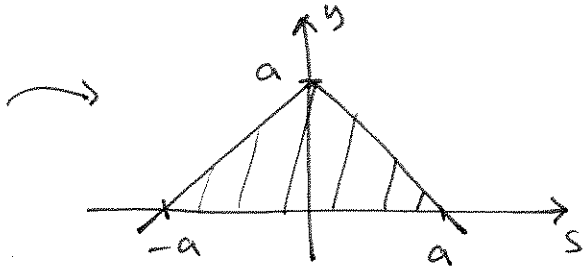
$$\therefore \int_0^3 e^x \, dx = e^3 - 1$$

5.4.10 Beräkna $\int_{-a}^{a} (a - |s|) \, ds$ genom att tolka integralen som en area.

Lösn.:







$$\Rightarrow \int_{-a}^{a} (a - |s|) \, ds = 2a \cdot a/2 = a^2 \text{ a.e.}$$

5.4.12 Beräkna $\int_0^2 \sqrt{2x - x^2} \, dx$ genom att tolka integralen som en area.

Lösn.: Hur ser grafen till $f(x) = \sqrt{2x - x^2}$ ut?

$$y = f(x) \Leftrightarrow y = \sqrt{2x - x^2} \Rightarrow y^2 = 2x - x^2 \Leftrightarrow x^2 + y^2 - 2x = 0$$

$$\Leftrightarrow \underbrace{x^2 - 2x + 1^2}_{(x-1)^2} - 1^2 + y^2 = 0 \Leftrightarrow (x-1)^2 + y^2 = 1^2$$

Cirkel med radie 1 och centrum (1, 0)

$$\therefore \int_0^2 \sqrt{2x - x^2}\, dx = \frac{1}{2} \cdot \pi \cdot 1^2 = \frac{\pi}{2} \text{ a.e.}$$