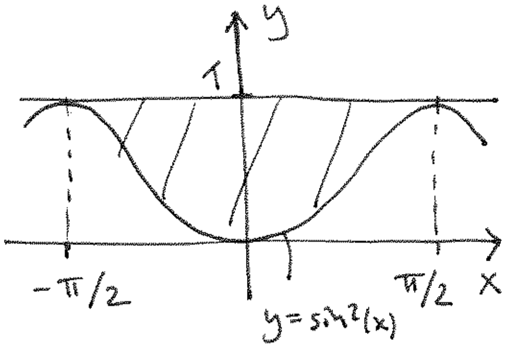
Envariabelsanalys Z & TD, ht2017, Räkneövning 2.2

5.7.18 Beräkna arean av området som begränsas av två efterföljande skärningar mellan kurvorna $y = \sin^2(x)$ och $y = 1$.



Lösn.: $\text{Area} = \int_{-\pi/2}^{\pi/2} (1 - \sin^2(x))\, dx =$

$$= \int_{-\pi/2}^{\pi/2} \cos^2(x)\, dx =$$

$$= \left\{ \begin{array}{l} \cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 \\ \Leftrightarrow \cos^2 x = \frac{1}{2}(1 + \cos(2x)) \end{array} \right\} =$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} 1 + \cos(2x)\, dx = \frac{1}{2}\left[x + \cancel{\frac{\sin(2x)}{2}} \right]_{-\pi/2}^{\pi/2} =$$

$$= \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{2} \right) = \frac{\pi}{2} \text{ a.e.}$$

6.1.7 Beräkna $\int \arctan(x)\, dx$

Lösn.: $\int \arctan(x)\, dx = \int 1 \cdot \arctan(x)\, dx =$

$$= x\arctan(x) - \int x \cdot \frac{1}{1+x^2}\, dx =$$

$$= x\arctan(x) - \frac{1}{2}\int \frac{d}{dx} \ln|1+x^2|\, dx =$$

$$= x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C$$

6.1.19 Beräkna $\int \cos(\ln(x))\, dx$

Lösn.: $\int \cos(\ln x)\,dx = \int 1\cdot\cos(\ln(x))\,dx =$

$= x\cos(\ln x) + \int x\,(+\sin(\ln(x)))\cdot\frac{1}{x}\,dx =$

$= x\cos(\ln x) + \int \sin(\ln(x))\,dx =$

$= x\cos(\ln x) + x\sin(\ln(x)) - \int \cancel{x}\cos(\ln(x))\cdot\frac{1}{\cancel{x}}\,dx$

$\Leftrightarrow 2\int\cos(\ln x)\,dx = x\left(\cos(\ln(x)) + \sin(\ln(x))\right)$

$\therefore \int\cos(\ln(x))\,dx = \frac{1}{2}x\left(\cos(\ln(x)) + \sin(\ln(x))\right) + C$

6.2.11 Beräkna $\int \frac{x-2}{x^2+x}\,dx$

Lösn.: $\frac{x-2}{x^2+x} = \frac{x-2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \Leftrightarrow$

$\Leftrightarrow x-2 = A(x+1) + Bx$

$x=0$: $-2 = A$

$x=-1$: $-3 = -B \Leftrightarrow B = 3$

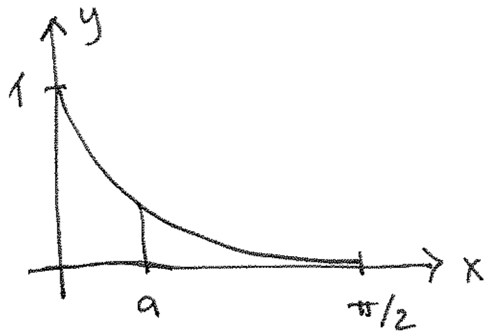
$\Rightarrow \int\frac{x-2}{x^2+x}\,dx = \int\left(-\frac{2}{x} + \frac{3}{x+1}\right)dx =$

$= -2\ln|x| + 3\ln|x+1| + C$

Bonus: Området under kurvan

$y = \frac{\cos(x)}{1+7\sin(x)}$, $0 \leq x \leq \frac{\pi}{2}$

delas av linjen $x=a$ i två delar vars areor förhåller sig



som 2:1 (den största arean till vänster).
Bestäm talet $a \in \mathbb{R}$.

Lösn.: Vill hitta $a \in \mathbb{R}$ s.a.

$$\int_0^a \frac{\cos(x)}{1+7\sin(x)}\,dx = 2\int_a^{\pi/2} \frac{\cos(x)}{1+7\sin(x)}\,dx \qquad (*)$$

$$\int_0^a \frac{\cos(x)}{1+7\sin(x)}\,dx = \left\{ \begin{array}{l} t=\sin(x), \quad x=0 \Leftrightarrow t=0 \\ dt=\cos(x)dx, \quad x=a \Leftrightarrow t=\sin(a) \end{array} \right\} =$$

$$= \int_0^{\sin(a)} \frac{1}{1+7t}\,dt = \frac{1}{7}\Big[\ln|1+7t|\Big]_0^{\sin(a)} =$$

$$= \frac{1}{7}\ln(1+7\sin(a))$$

$$\int_a^{\pi/2} \frac{\cos(x)}{1+7\sin(x)}\,dx = \left\{ \begin{array}{l} t=\sin(x), \quad x=a \Leftrightarrow t=\sin(a) \\ dt=\cos(x)dx, \quad x=\frac{\pi}{2} \Leftrightarrow t=1 \end{array} \right\} =$$

$$= \int_{\sin(a)}^{1} \frac{1}{1+7t}\,dt = \frac{1}{7}\Big[\ln|1+7t|\Big]_{\sin(a)}^{1} =$$

$$= \frac{1}{7}\Big(\ln(8) - \ln(1+7\sin(a))\Big)$$

Insatt i (*) ger detta:

$$\frac{1}{\cancel{7}}\ln(1+7\sin(a)) = \frac{2}{\cancel{7}}\Big(\ln(8) - \ln(1+7\sin(a))\Big)$$

$$\Leftrightarrow 3\ln(1+7\sin(a)) = 2\ln(8) \Leftrightarrow$$

$$\Leftrightarrow \ln\left((1+7\sin(a))^3\right) = \ln(8^2) \Leftrightarrow$$

$$\Leftrightarrow (1+7\sin(a))^3 = 64 \Leftrightarrow$$

$$\Leftrightarrow 1+7\sin(a) = (4^3)^{1/3} = 4 \Leftrightarrow$$

$\Leftrightarrow 7\sin(a) = 3$

$$\therefore \; a = \arcsin\left(\frac{9}{7}\right)$$